Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 5, 1995, 291–304

p-REGULAR MAPPINGS AND ALTERNATIVE RESULTS FOR PERTURBATIONS OF m-ACCRETIVE OPERATORS IN BANACH SPACES

Zouhua Ding — Athanassios G. Kartsatos

Dedicated to Ky Fan

1. Introduction

In what follows, the symbol X stands for a real Banach space with norm $\|\cdot\|$ and (normalized) duality mapping J. Moreover, "continuous" means "strongly continuous" and the symbol "$\to$" ("$\rightharpoonup$") means strong (weak) convergence. The symbol $\mathbb{R}$ ($\mathbb{R}_+$) stands for the set $(-\infty,\infty)$ ($[0,\infty)$) and the symbols ∂D, $\operatorname{int} D$, $\overline{D}$ denote the strong boundary, interior and closure of the set D, respectively. An operator $T : X \supset D(T) \to Y$, with Y another real Banach space, is *bounded* if it maps bounded subsets of $D(T)$ onto bounded sets of Y. It is *compact* if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets of Y. It is called *demicontinuous* (*completely continuous*) if it is strong-weak (weak-strong) continuous on $D(T)$. For a multi-valued operator $T : X \to 2^X$ and any set $A \subset X$, we set $D(T) = \{x \in X : Tx \neq \emptyset\}$ and $TA = \bigcup\{Tx : x \in A\}$ and we always assume that $D(T) \neq \emptyset$. An operator $T : X \supset D(T) \to 2^X$ is *accretive* if for every $x, y \in D(T)$ there exists $j \in J(x-y)$ such that

$$(*) \qquad \langle u - v, j\rangle \geq 0 \qquad \text{for every } u \in Tx,\ v \in Ty.$$

1991 *Mathematics Subject Classification.* Primary 47H17; Secondary 47H05, 47H10.
Key words and phrases. Accretive operator, m-accretive operator, compact perturbation, compact resolvent, p-regular mapping, regular mapping, alternative theorem.

An accretive operator T is *strongly accretive* if 0 in the right-hand side of $(*)$ is replaced by $\alpha\|x-y\|^2$, where $\alpha > 0$ is a fixed constant. An accretive operator T is called *m-accretive* if $R(T+\lambda I) = X$ for every $\lambda > 0$, where I denotes the identity operator on X.

We denote by $B_r(0)$ the open ball of X with center at zero and radius $r > 0$. For an m-accretive operator T, the *resolvents* $J_\lambda : X \to D(T)$ of T are defined by $J_\lambda = (I+\lambda T)^{-1}$ for all $\lambda \in (0,\infty)$ and are nonexpansive mappings (i.e., Lipschitz continuous with Lipschitz constant 1). An operator $T : X \supset D(T) \to 2^X$ is *ϕ-expansive* on $D \subset X$ if there exists a strictly increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(0) = 0$ and for every $x, y \in D(T) \cap D$ and every $u \in Tx$, $v \in Ty$ we have

$$\|u - v\| \geq \phi(\|x-y\|).$$

If T is ϕ-expansive on $D(T)$, then we say that T is just ϕ-expansive. A ϕ-expansive operator is called *c-expansive* $(c > 0)$ if we can choose the function ϕ so that $\phi(u) \equiv cu,\ u \in \mathbb{R}_+$. Let $\mathcal{B}$ denote the family of all bounded subsets of the space X. The *Kuratowski measure of noncompactness* is a function $\gamma : \mathcal{B} \to \mathbb{R}_+$ defined by

$$\gamma(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by a finite family of sets of diameter } < \varepsilon\}.$$

The Kuratowski measure γ has the following properties. We assume that $A, B \in \mathcal{B}$.

(i) $\gamma(A) = 0$ if and only if $\bar{A}$ is compact;
(ii) $\gamma(\overline{\text{co}}\, A) = \gamma(A)$, where $\overline{\text{co}}\, A$ denotes the closed convex hull of the set A;
(iii) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$;
(iv) $\gamma(tA) = |t|\gamma(A)$ for every $t \in \mathbb{R}$;
(v) $\gamma(A+B) \leq \gamma(A) + \gamma(B)$.

Given a continuous operator $T : X \supset D(T) \to X$ and $k \geq 0$, we say that T is *k-set-contractive* if for every bounded $A \subset D(T)$ we have $\gamma(T(A)) \leq k\gamma(A)$. Naturally, this definition makes sense only if $T(A) \in \mathcal{B}$ for every bounded $A \subset D(T)$. It is well known that if $T_1 : X \supset D(T_1) \to X$ is a k_1-set-contraction, $T_2 : D(T_1) \to X$ a k_2-set-contraction and $T_3 : R(T_1) \to X$ a k_3-set-contraction, then $T_1 + T_2 : D(T_1) \to X$ is a (k_1+k_2)-set-contraction and $T_3 \circ T_1 : D(T_1) \to X$ is a k_1k_3-set-contraction. Important examples of k-set-contractions with $k < 1$ are mappings of the type $T = S + C : X \supset D(T) \to X$, where S is a *strict contraction* ($\|Sx - Sy\| \leq k\|x-y\|,\ x, y \in D(T)$) and $C : D(T) \to X$ is a compact map. For convenience, we say that the operator $T : X \supset D(T) \to X$ is a *γ-contraction* if it is a k-set-contraction with $k < 1$.

We say that a continuous operator $T : X \supset D(T) \to X$ is *condensing* if for every nonempty, bounded, noncompact set $A \subset D(T)$ with $\gamma(A) > 0$ we have $\gamma(T(A)) < \gamma(A)$. It is obvious that every k-set-contraction with $k < 1$ is condensing, but the converse is not true in general. Nussbaum has shown the following result (cf. Petryshyn [24]):

Lemma A. *Let $D \subset X$ be closed, convex and bounded and $T : D \to D$ condensing. Then T has a fixed point in D.*

For facts involving accretive operators, and other related concepts, the reader is referred to Barbu [1], Browder [2], Ciorănescu [5] and Lakshmikantham and Leela [20]. A survey article on compact perturbations and compact resolvents of accretive operators can be found in [19].

The purpose of this paper is to initiate the study of p-regular mappings. The concept of a p-regular mapping is an extension of the concept of an essential mapping introduced by Granas in [12]. It is also an extension of the concept of a p-0-epi mapping introduced by Furi, Martelli and Vignoli in [9]. As the authors of [9] and [21] have pointed out, the study of such mappings allows us to obtain existence results for various types of operator equations $Tx + Cx = 0$, involving set-contractions C, without using any type of degree theory. Other results on p-0-epi mappings can be found in Furi and Pera [10] and Pera [23]. On the other hand, alternative results involving sums of two operators can be found in Chang [3] ($T = I$ and C is nonexpansive), Dugundji and Granas [8] ($T = I$ and C is a k-set-contraction), and Górniewicz and Kucharski [11], where T is a Vietoris mapping and CT^{-1} is a set-contraction.

In Section 2 we introduce the concept of a p-regular mapping and apply such regularity considerations to inclusions involving multi-valued m-accretive, L-expansive operators. In Section 3 we show how one may apply the results of Section 2 in order to obtain alternative results for such inclusions. Theorem 2 of Section 3 is the main alternative result of the paper involving m-accretive, but not necessarily L-expansive, operators T. In Section 4 we show the compactness, or the weak compactness, of the set of solutions of such inclusions and in Section 5 we give an example of a partial differential equation to which our theory can be applied. Our methods are mainly extensions of the methods used in [9] and [21].

2. p-Regular mappings and m-accretive operators

Definition 1. Let $G \subset X$ be open and bounded and let $T : X \supset D(T) \to 2^X$ be such that $D(T) \cap G \neq \emptyset$ and $Tx \not\ni 0$, $x \in D(T) \cap \partial G$. We say that T is *p-regular* on G if for every continuous p-set-contraction $h : \overline{G} \to X$, vanishing everywhere on ∂G, we have $Tx \ni h(x)$ for at least one $x \in D(T) \cap G$. We also use the term *regular* for 0-regular operators.

We note that if T is p-regular and $q \in [0, p)$, then T is q-regular. Our definition of p-regularity is more general than the definition of a p-0-epi mapping of Martelli [19] and other authors mentioned therein. The operator T is now a multi-valued operator defined on an arbitrary set.

If the operator $T : X \supset D(T) \to 2^X$ is L-expansive, then it is easy to see that $Tx \cap Ty \neq 0$ implies that $x = y$ and, naturally, $Tx = Ty$.

Lemma 1. *Let $T : X \supset D(T) \to 2^X$ be m-accretive and L-expansive. Let G be open, bounded and such that $D(T) \cap G \neq \emptyset$. Then for every $y_0 \in T(D(T) \cap G)$ and every $\varepsilon \in (0, L)$ the mapping $Tx - y_0$ is $(L - \varepsilon)$-regular on G.*

Proof. Since $y_0 \in T(D(T) \cap G)$ and the operator T is L-expansive, we have $T(D(T) \cap \partial G) - y_0 \not\ni 0$. In fact, we know that $y_0 = T(x)$ for some $x \in D(T) \cap G$. If we also have $y_0 = T(y)$, for some $y \in T(D(T) \cap \partial G)$, then $x = y$, which contradicts the fact that $G \cap \partial G = \emptyset$.

It is known that T is surjective with a Lipschitz continuous inverse $T^{-1} : X \to D(T)$. To see the surjectivity of T, fix $p \in X$ and let x_n solve $Tx + (1/n)x \ni p$. Then $\{x_n\}$ is a bounded sequence. In fact, assuming, without loss of generality, that $\|x_n\| \to \infty$, we obtain, for some $u_n \in Tx_n$,

$$\liminf_{n\to\infty} \frac{\|u_n\|}{\|x_n\|} \leq \lim_{n\to\infty} \left[\frac{1}{n} + \frac{\|p\|}{\|x_n\|}\right] = 0.$$

However, this contradicts

$$\liminf_{n\to\infty} \frac{\|u_n\|}{\|x_n\|} \geq L > 0,$$

which follows from the L-expansiveness of T. Since $\{x_n\}$ is bounded, we have $(1/n)x_n \to 0$ as $n \to \infty$. Thus, for some $u_n \in Tx_n$,

$$L\|x_n - x_m\| \leq \|u_n - u_m\| \leq \|(1/n)x_n - (1/m)x_m\| \to 0 \qquad \text{as } m, n \to \infty.$$

Since $x_n \to$ (some) $x_0 \in X$, $u_n \to p$ and T is closed, we have $x_0 \in D(T)$ and $Tx_0 = p$.

Let $\varepsilon \in (0, L)$ be given and let $h : \overline{G} \to X$ be a continuous $(L - \varepsilon)$-set-contraction such that $h(x) = 0$ for $x \in \partial G$. Choose $r > 0$ so that

$$r \geq \|T^{-1}(h(x) + y_0)\|, \qquad x \in \overline{G}.$$

This is possible because $h(\overline{G})$ is bounded and T^{-1} is Lipschitz continuous, and thus bounded, with Lipschitz constant $1/L$. We define the mapping $h_1 : X \to X$ as follows:

$$h_1(x) = \begin{cases} T^{-1}(h(x) + y_0), & x \in G, \\ T^{-1}y_0, & x \notin G. \end{cases}$$

Since h and T^{-1} are continuous, it is easy to see that h_1 is continuous and such that its restriction $\overline{h}_1 : \overline{B_r(0)} \to \overline{B_r(0)}$ is a γ-contraction (and thus condensing) with constant $(L-\varepsilon)/L$. To see the latter, let $A \subset \overline{B_r(0)}$. Then $A = (A \cap G) \cup (A \cap (X \setminus G))$. Thus,

$$\begin{aligned}\gamma(h_1(A)) &= \max\{\gamma(h_1(A \cap G)), \gamma(h_1(A \cap (X \setminus G)))\}\\ &= \max\{\gamma(h_1(A \cap G)), \gamma(\{T^{-1}y_0\})\}\\ &= \gamma(T^{-1}(h(A \cap G) + y_0))\\ &\le [(L-\varepsilon)/L]\gamma(A \cap G)\\ &\le [(L-\varepsilon)/L]\gamma(A).\end{aligned}$$

By Lemma A, there exists a point $\overline{x} \in \overline{B_r(0)}$ such that $\overline{h}_1(\overline{x}) = \overline{x}$. If $\overline{x} \notin G$, then $\overline{x} = \overline{h}_1(\overline{x}) = T^{-1}y_0$. Since $y_0 \in T(D(T) \cap G)$, we have $\overline{x} = T^{-1}y_0 \in D(T) \cap G$, i.e., a contradiction. It follows that $\overline{x} \in G$, which implies $\overline{x} = T^{-1}(h(\overline{x}) + y_0)$. Thus, $\overline{x} \in D(T) \cap G$ and $T\overline{x} - y_0 \ni h(\overline{x})$. We have shown that $Tx - y_0$ is $(L-\varepsilon)$-regular on G. □

Lemma 1 leads to the following proposition which is the essence of the alternative results of Section 3.

Proposition 1. *Let $G \subset X$ be open and bounded. Let $T : X \supset D(T) \to 2^X$ be m-accretive and L-expansive with $D(T) \cap G \neq \emptyset$. Assume that $C : D(T) \to X$ is a p-set-contraction with constant $p \in [0, L)$. Let $y_0 \in X$, $\varepsilon \in (0, L-p)$ and assume that $Tx + tCx - y_0 \not\ni 0$, $t \in [0,1]$, $x \in D(T) \cap \partial G$. Then*

(i) *if $y_0 \notin T(D(T) \cap G)$, the operator $Tx + Cx - y_0$ is not p-regular on G;*
(ii) *if $y_0 \in T(D(T) \cap G)$, the operator $Tx + Cx - y_0$ is $(L-p-\varepsilon)$-regular on G.*

Proof. Let $y_0 \notin T(D(T) \cap G)$. Then, by our hypothesis, $y_0 \notin T(D(T) \cap \overline{G})$. Since $T^{-1} : X \to D(T)$ is continuous, the set $T(D(T) \cap \overline{G})$ is closed, being the image of a closed set in the relative topology of $D(T)$. Similarly, the set $T(D(T) \cap G)$ is open. Thus,

$$\delta = \inf\{\|Tx - y_0\| : x \in D(T) \cap \overline{G}\} > 0.$$

We choose $\varrho \in (0,1)$ so that

$$\varrho\|Cx\| < \delta, \qquad x \in D(T) \cap \overline{G}.$$

Let us assume that $Tx + Cx - y_0$ is p-regular. Then the set S_1, defined by

$$S_1 = \{x \in D(T) \cap G : Tx + tCx - y_0 \ni 0 \text{ for some } t \in [0,1]\}$$

is nonempty and compact. In fact, $S_1 \neq \emptyset$ because $y_0 \in (T+C)(D(T) \cap G)$ (take $t = 1$, $h \equiv 0$ in Definition 1, where T is appropriately replaced by $T + C$). To show the compactness of S_1, we observe that

$$TS_1 = \{u \in T(D(T) \cap G) : u = -tCT^{-1}u + y_0 \text{ for some } t \in [0,1]\},$$

which implies

$$\gamma(TS_1) \le t\gamma(CT^{-1}(TS_1)) + \gamma(\{y_0\}) \le (p/L)\gamma(TS_1).$$

This says that $\gamma(TS_1) = 0$, i.e., that TS_1 is relatively compact. To show that TS_1 is closed, let $\{u_n\} \subset TS_1$ be such that $u_n \to u_0 \in X$. Then $u_n \in T(D(T) \cap G)$,

$$u_n + t_n CT^{-1}u_n - y_0 = 0$$

for some sequence $\{t_n\} \subset [0,1]$, and $u_0 \in \overline{T(D(T) \cap G)}$. Let $u_n \in Tx_n$, where $x_n \in D(T) \cap G$. Then $x_n = T^{-1}u_n \to T^{-1}u_0 \equiv \overline{x} \in \overline{D(T) \cap G}$. Since T is closed, being m-accretive, $\overline{x} \in D(T)$ and $u_0 \in T\overline{x}$. Thus, $u_0 \in T(D(T) \cap \overline{G})$ and, assuming that $t_n \to t_0 \in [0,1]$,

$$u_0 + t_0 CT^{-1}u_0 - y_0 = 0.$$

This says that

$$T\overline{x} + t_0 C\overline{x} - y_0 \ni 0,$$

where $\overline{x} \in D(T) \cap \overline{G}$. However, our assumption implies that $\overline{x} \in D(T) \cap G$, i.e., $u_0 \in T(D(T) \cap G)$. It follows that $u_0 \in TS_1$, i.e., TS_1 is closed and thus compact. Since $S_1 = T^{-1}(TS_1)$, we have the compactness, and thus the closedness, of S_1. By Urysohn's lemma, there exists a continuous function $\phi : X \to [0,1]$ such that

$$\phi(x) = \begin{cases} 1, & x \in S_1, \\ 0, & x \in \partial G. \end{cases}$$

We set

$$g(x) \equiv (1-\varrho)\phi(x)Cx.$$

We see that $g(x) = 0,\ x \in \partial G$, and that g is a $(1-\varrho)p$-set-contraction. Since $(1-\varrho)p < p$, the operator $(T + C)x - y_0$ is $(1-\varrho)p$-regular. It follows that the inclusion $Tx + Cx - y_0 \ni g(x)$ must have a solution, i.e., there exists $x \in D(T) \cap G$ such that

$$Tx + [1 - (1-\varrho)\phi(x)]Cx - y_0 \ni 0.$$

Since $0 \le 1 - (1-\varrho)\phi(x) \le 1$, we conclude that $x \in S_1$, which implies that $\phi(x) = 1$. Consequently, $Tx + \varrho Cx - y_0 \ni 0$, or $-\varrho Cx \in Tx - y_0$. However, $\varrho\|Cx\| < \delta$ yields the desired contradiction. This completes the proof of the fact that $Tx + Cx - y_0$ is not p-regular whenever $y_0 \notin T(D(T) \cap G)$.

To show the second part of the theorem, we assume that $y_0 \in T(D(T) \cap G)$ and let $h : \overline{G} \to X$ be an $(L - p - \varepsilon)$-contraction such that $h(x) = 0,\ x \in \partial G$. We define the set

$$S_2 = \{x \in D(T) \cap G : Tx + tCx - y_0 \ni h(x) \text{ for some } t \in [0,1]\}$$

and note that $S_2 \ne \emptyset$ because $Tx - y_0$ is $(L - \varepsilon)$-regular by Lemma 1. Also, from

$$TS_2 = \{u \in T(D(T) \cap G) : u = -tCT^{-1}u + y_0 - h(T^{-1}u) \text{ for some } t \in [0,1]\}$$

and

$$\begin{aligned}\gamma(TS_2) &\le t\gamma(-CT^{-1}(TS_2)) + \gamma(h(T^{-1}(TS_2)))\\ &\le t(p/L)\gamma(TS_2) + [(L-p-\varepsilon)/L]\gamma(TS_2)\\ &\le [(L-\varepsilon)/L]\gamma(TS_2),\end{aligned}$$

we conclude that $\gamma(TS_2) = 0$, which shows the relative compactness of the set TS_2. Working as before, we can also see that TS_2 is closed. Thus, TS_2 is compact and so is $S_2 = T^{-1}(TS_2)$. Using again Urysohn's lemma, we construct a function ϕ as above and consider the inclusion

$$Tx - y_0 \ni -\phi(x)Cx + h(x).$$

This inclusion has a solution $x \in D(T) \cap G$ because the mapping $-\phi(x)Cx + h(x)$ is $(L-\varepsilon)$-set-contractive. In fact, this mapping is continuous and, for $A \subset \overline{G}$,

$$\begin{aligned}\gamma(-\phi(A)CA + h(A)) &\le \gamma(-\phi(A)CA) + \gamma(h(A))\\ &= \gamma(\phi(A)CA) + \gamma(h(A))\\ &\le [\max_{t\in\phi(A)}\{t\}]\gamma(CA) + \gamma(h(A))\\ &\le \gamma(CA) + \gamma(h(A))\\ &< [p + (L-p-\varepsilon)]\gamma(A) = (L-\varepsilon)\gamma(A).\end{aligned}$$

Here we have used Remark 1.4.1 in Lakshmikantham and Leela [20]. Thus, for some $x \in D(T) \cap G$, we have $Tx + \phi(x)Cx - y_0 \ni h(x)$. Again, we must have $x \in S_2$ and $\phi(x) = 1$. Consequently, $Tx + Cx - y_0 \ni h(x)$, and we have the proof that $Tx + Cx - y_0$ is $(L-p-\varepsilon)$-regular whenever $y_0 \in T(D(T) \cap G)$. □

For compact mappings C, we have the following important corollary.

Corollary 1. *Let $G \subset X$ be open and bounded. Let $T : X \supset D(T) \to 2^X$ be m-accretive and L-expansive with $D(T) \cap G \neq \emptyset$. Assume that $C : D(T) \to X$ is compact. Let $y_0 \in X$, $\varepsilon \in (0, L)$ and assume that $Tx + tCx - y_0 \not\ni 0$, $t \in [0,1]$, $x \in D(T) \cap \partial G$. Then*

(i) *if $y_0 \notin T(D(T) \cap G)$, the operator $Tx + Cx - y_0$ is not regular on G;*
(ii) *if $y_0 \in T(D(T) \cap G)$, the operator $Tx + Cx - y_0$ is $(L-\varepsilon)$-regular on G.*

Proof. Just take $p = 0$ in Proposition 1. □

3. Alternative results

We are now ready for the first alternative statement involving set-contractive perturbations of an m-accretive, L-expansive operator T.

Theorem 1. *Let $G \subset X$ be open and bounded. Let $T : X \supset D(T) \to 2^X$ be m-accretive and L-expansive with $0 \in T(D(T) \cap G)$. Let $C : D(T) \to X$ be*

p-set-contractive with constant $p \in [0, L)$, and let $\varepsilon \in (0, L - p)$. Then at least one of the following statements holds:

(i) *the inclusion $Tx + Cx \ni h(x)$ has a solution $x \in D(T) \cap G$ for every $(L - p - \varepsilon)$-set-contraction h vanishing identically on ∂G. In particular, there exists $x \in D(T) \cap G$ such that $Tx + Cx \ni 0$;*

(ii) *there exist $x \in D(T) \cap \partial G$ and $\lambda \in (0, 1]$ such that $Tx + \lambda Cx \ni 0$.*

Proof. We assume that $Tx + \lambda Cx \not\ni 0$ for every $x \in D(T) \cap \partial G$, $\lambda \in (0, 1]$ and show (i). We observe that $0 \in T(D(T) \cap G)$ and the L-expansiveness of T preclude T from having another zero in $D(T) \cap \partial G$. Thus, $Tx + \lambda Cx \neq 0$ for every $x \in D(T) \cap \partial G$ and every $\lambda \in [0, 1]$. Since $0 \in T(D(T) \cap G)$, we may apply Proposition 1, with $y_0 = 0$, in order to conclude that the operator $T + C$ is $(L - p - \varepsilon)$-regular. This completes the proof. □

It should be noted that the above theorem does not follow from the "condensing" versions of the results of Chen in [4], whenever C is condensing and L, p are appropriately chosen. Unfortunately, Chen's degree theory is not valid for condensing mappings C as claimed in [4, p. 403]. The reason for this is that the mapping $Q_\lambda \equiv (T + \lambda I)^{-1}$ is not generally nonexpansive as Chen claims in [4, p. 394]. In fact,

$$Q_\lambda(x) = \left(\frac{1}{\lambda}T + I\right)^{-1}\left(\frac{1}{\lambda}x\right),$$

which says that Q_λ is Lipschitz continuous with Lipschitz constant $1/\lambda$. Thus, it is not possible to obtain condensing mappings of the type $(T + \lambda I)^{-1}C$ for all small $\lambda > 0$, unless C is a compact operator. Some corrections in the calculations of Chen [4] are thus in order. For example, the calculations on pages 396–397 there need appropriate adjustments.

The next alternative theorem involves compact perturbations of m-accretive operators. We denote by $\operatorname{co} A$ the convex hull of the set A.

Theorem 2. *Let $G \subset X$ be open and bounded. Let $T : X \supset D(T) \to 2^X$ be m-accretive with $0 \in D(T) \cap G$ and $0 \in T(0)$. Let T be ϕ-expansive on ∂G and $C : D(T) \to X$ compact. Then at least one of the following statements holds:*

(i) *for every compact function $h : \overline{G} \to X$ vanishing identically on ∂G we have $\overline{(T + C - h)(D(T) \cap G)} \ni 0$;*

(ii) *there exists $x \in D(T) \cap \partial G$ and $\lambda \in [0, 1]$ such that $Tx + \lambda Cx \ni 0$.*

If, moreover, X is uniformly convex and $C : \overline{D(T)} \to X$ is completely continuous, then (i) *can be replaced by*

(ia) *there exists $x \in D(T) \cap \overline{\operatorname{co} G}$ such that $Tx + Cx \ni h(x)$, where $h : \overline{\operatorname{co} G} \to X$ is a completely continuous mapping vanishing identically on ∂G.*

Proof. As in the proof of Theorem 1, we may assume that $Tx+\lambda Cx \not\ni 0$ for every $x \in D(T) \cap \partial G$, $\lambda \in [0,1]$, to show the inclusion $\overline{(T+C-h)(D(T)\cap G)} \ni 0$. We show first that the inclusion

$$Tx + \lambda Cx + (1/n)x \ni 0$$

has no solution in $D(T) \cap \partial G$, for all $\lambda \in [0,1]$ and all large n. In fact, assuming that this is not true, we may also assume that there exists a sequence $\{\lambda_n\} \subset [0,1]$ and a sequence $\{x_n\} \subset D(T) \cap \partial G$ such that

$$Tx_n + \lambda_n Cx_n + (1/n)x_n \ni 0.$$

Since $\{x_n\}$ lies in a bounded set, we may assume that $Cx_n \to y \in X$. We may also assume that $\lambda_n \to \lambda_0 \in [0,1]$. Since T is ϕ-expansive on ∂G, it follows that $\{x_n\}$ is a Cauchy sequence. Letting $x_n \to x_0$ and using the closedness of the operator T, we deduce that $x_0 \in D(T) \cap \partial G$ and $Tx_0 + \lambda_0 Cx_0 \ni 0$. This contradiction shows that the inclusion (i) has no solution on $D(T) \cap \partial G$ for all large n. We may assume that this happens for all n. Using Corollary 1 (with $y_0 = 0$ and $0 \in (T+(1/n)I)(0)$, we see now the mapping $Tx+Cx+(1/n)x$ is $[(1/n)-\varepsilon_n]$-regular, where $\varepsilon_n \in (0,1/n)$. As such it is also regular, i.e., $Tx+Cx+(1/n)x \ni h(x)$ has a solution x_n in $D(T)\cap G$ for every $n = 1,2,\dots$, where $h : \overline{G} \to X$ is a compact function vanishing identically on ∂G. Since $x_n/n \to 0$ as $n \to \infty$, it follows that $0 \in \overline{R(T+C-h)}$.

The second part of the theorem follows as in Lemma 2 of Kartsatos [17], or more generally, Lemma 1 of Guan and Kartsatos [13]. In fact, since X is reflexive, C is also compact and we may assume that $x_n \rightharpoonup x_0 \in X$. By that lemma, we have $Tx_0 + Cx_0 \ni h(x_0)$. Naturally, $x_0 \in D(T) \cap \overline{\mathrm{co}\, G}$. □

4. Compactness of the solution set

It is easy to see that if $G \subset X$ is open and bounded and $C : \overline{G} \to X$ is compact, then the solution set of the equation $(I+C)(x) = 0$ is compact. It is thus interesting to see whether the relevant problem for the inclusion $Tx+Cx \ni 0$ has a similar answer. To this end, we give below a lemma in this direction, which is inspired by the proof of Proposition 1.

Theorem 3. *Let $G \subset X$ be open and bounded. Let $T : X \supset D(T) \to 2^X$ be m-accretive and L-expansive with $D(T) \cap G \neq \emptyset$. Assume that $C : D(T) \to X$ is a p-set-contraction with constant $p \in [0,L)$. Fix $y_0 \in X$ and assume that $Tx + tCx - y_0 \not\ni 0$, $t \in [0,1]$, $x \in D(T) \cap \partial G$. Then if $y_0 \in T(D(T) \cap G)$, the solution set*

$$S \equiv \{x \in D(T) \cap G : Tx + Cx - y_0 \ni 0\}$$

is nonempty and compact.

Proof. By the conclusion of Proposition 1, the operator $T + C - y_0$ is $(L - p - \varepsilon)$-regular for any $\varepsilon \in (0, L - p)$. In particular, it is regular. Thus, the equation $Tx + Cx - y_0 \ni 0$ has at least one solution in $D(T) \cap G$. This says that S is nonempty. Its compactness follows as in the case of the compactness of the set S_2 in the proof of Proposition 1. □

The next theorem shows the weak compactness of the solution set in Theorem 2, provided that G is convex, X is uniformly convex and C is completely continuous.

Theorem 4. *Let the assumptions of Theorem 2 be satisfied with X uniformly convex, the set G convex and $C : \overline{D(T)} \to X$ completely continuous. Assume that $Tx + \lambda Cx \not\ni 0$ for every $x \in D(T) \cap \partial G$, $\lambda \in [0, 1]$. Then the set*

$$S \equiv \{x \in D(T) \cap G : Tx + Cx - y_0 \ni 0\}$$

is nonempty and weakly compact.

Proof. The fact that S is nonempty follows from Theorem 2. To show that S is weakly sequentially compact, assume for convenience that $y_0 = 0$ and let $\{x_n\} \subset S$. Then, since X is reflexive, there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$x_n \rightharpoonup x_0 \in \overline{\mathrm{co}}\,(D(T) \cap G) \subset \overline{\mathrm{co}}\,(D(T)) \cap \overline{\mathrm{co}}\,G = \overline{D(T)} \cap \overline{G}.$$

(We have used above the fact that $\overline{D(T)}$ is convex. This can be found in Barbu [1, Proposition 3.6] and Ciorănescu [5, Theorem 1.15]. However, the uniform convexity of X^* was never used in either one of these two references.) Thus,

$$Tx_n + Cx_n + (1/n)x_n \ni (1/n)x_n, \qquad n = 1, 2, \ldots\,,$$

or

$$Tx_n + Cx_n + \alpha_n x_n \ni p_n, \qquad n = 1, 2, \ldots\,,$$

where α_n, p_n are obviously defined. By Lemma 2 of [17] or Lemma 1 of [13], we conclude that $x_0 \in D(T) \cap \overline{G}$ and $Tx_0 + Cx_0 \ni 0$. Since, by our assumption, $x_0 \notin D(T) \cap \partial G$, we see that $x_0 \in D(T) \cap G$, i.e., $x_0 \in S$. We have shown that S is weakly sequentially compact. By the Eberlein–Shmul'yan theorem, S is weakly compact and the proof is complete. □

5. Discussion and example

We consider an application to a partial differential equation from Massabò and Stuart [20]:

$$-\Delta u(x) + q(x)u(x) + b(x, u(x), \nabla u(x)) = 0, \qquad x \in \mathbb{R}^n,$$

where $n > 2$. We make the following assumptions.

(1) $q : \mathbb{R}^n \to \mathbb{R}$ is continuous and

$$0 < \inf_{x \in \mathbb{R}^n} q(x) \le \sup_{x \in \mathbb{R}^n} q(x) < \infty.$$

(2) $b : \mathbb{R}^{2n+1} \to \mathbb{R}$ is continuous and satisfies the following two conditions.

(2a) There exist constants $p \in [1, n/(n-2))$, $c \in \mathbb{R}_+$ and a continuous function $g \in L^2(\mathbb{R}^n)$ such that

$$|b(x, \eta)| \le g(x) + c\|\eta\|^p, \qquad (x, \eta) \in \mathbb{R}^n \times \mathbb{R}^{n+1},$$

where $\|\eta\|$ denotes the Euclidean norm of η.

(2b) For every $\varepsilon > 0$ there exist constants $p = p(\varepsilon) \in [1, n/(n-2))$ and $l = l(\varepsilon) \ge 0$ such that

$$|b(x, 0) - b(x, \eta)| \le \varepsilon\|\eta\|^p$$

for every $x \in \mathbb{R}^n$ with $\|x\| \ge l$ and all $\eta \in \mathbb{R}^{n+1}$.

The operators $T : W^{2,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $C : W^{2,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are defined by $(Tu)(x) \equiv -\Delta u(x) + q(x)u(x)$ and $(Cu)(x) \equiv b(x, u(x), \nabla u(x))$, respectively. The operator T is self-adjoint, m-accretive, strongly accretive, and such that $T^{-1} : L^2(\mathbb{R}^n) \to W^{2,2}(\mathbb{R}^n)$ is a Q^{-1}-set-contraction. Here,

$$Q \equiv \inf \sigma_{\mathrm{e}}(T) \in (0, \infty),$$

where $\sigma_{\mathrm{e}}(T)$ is the essential spectrum of T. As Massabò and Stuart have shown in [20], the operator C is compact. It follows that the alternative result of Theorem 1 applies here for a family of appropriate sets G because T is strongly accretive, and thus L-expansive, on the entire space $W^{2,2}(\mathbb{R}^n)$. In particular, letting $G = B_r(0) \subset L^2(\mathbb{R}^n)$ for some $r > 0$, we conclude that either there exists $u \in W^{2,2}(\mathbb{R}^n)$ with $\|u\|_{L^2(\mathbb{R}^n)} = r$ and $\lambda \in (0, 1]$ such that $Tu + \lambda Cu = 0$, or there exists $u \in B_r(0) \cap W^{2,2}(\mathbb{R}^n)$ such that $Tu + Cu = 0$.

It is possible to have general homotopy results for p-regular mappings in the spirit of [9]. We exhibit such a property below and then we give an application of it to the solvability of eigenvalue problems where the eigenvalue λ is not of multiplicative nature as in the alternative results of Section 3.

Theorem 5. *Let $T : X \supset D(T) \to 2^X$ be m-accretive and L-expansive. Let G be open, bounded and such that $D(T) \cap G \neq \emptyset$. Let $0 \in T(D(T) \cap G)$ and let $H : [a,b] \times \overline{G} \to X$ be compact and such that $H(0,x) = 0$, $x \in \partial G$, where a, b are constants with $a \leq 0 \leq b$. Assume that $Tx + H(t,x) \not\ni 0$ for every $(t,x) \in [a,b] \times (D(T) \cap \partial G)$. Then $T + H(\lambda, \cdot)$ is regular for every $\lambda \in [a,b]$.*

Proof. Fix $\lambda = \lambda_0 \in [a,b]$ and let $h : \overline{G} \to X$ be compact and such that $h(x) = 0$, $x \in \partial G$. We need to show that the inclusion $Tx + H(\lambda_0, x) \ni h(x)$ is solvable in $D(T) \cap G$. To this end, we examine the set

$$S \equiv \{x \in D(T) \cap G : Tx + H(t,x) \ni h(x) \text{ for some } t \in [a,b]\}$$

and its image

$$TS = \{u \in T(D(T) \cap G) : u = -H(t, T^{-1}u) + h(T^{-1}u) \text{ for some } t \in [a,b]\}.$$

As in the proof of Proposition 1, it can be seen that the set S is compact. By Urysohn's lemma, there exists a mapping $\phi : X \to [0,1]$ such that $\phi(S) = \{1\}$ and $\phi(\partial G) = \{0\}$. We let $\phi_1(x) \equiv \lambda_0 \phi(x)$. We observe that the mapping $H(\phi_1(x), x) - h(x)$ is compact and that it vanishes identically on the set ∂G. Since T is regular, by Lemma 1, the inclusion $Tx \ni -H(\phi_1(x), x) + h(x)$ is solvable for some $x_0 \in D(T) \cap G$. Since we must have $x_0 \in S$, we see that $\phi_1(x_0) = \lambda_0$, i.e., $Tx_0 + H(\lambda_0, x_0) \ni h(x_0)$. □

Corollary 2. *Let $T : X \supset D(T) \to 2^X$ be m-accretive and L-expansive. Let G be open, bounded and such that $D(T) \cap G \neq \emptyset$. Let $0 \in T(D(T) \cap G)$ and let $H : [0,1] \times \overline{G} \to X$ be compact and such that $H(0,x) = 0$, $x \in \partial G$. Then there exists $\varepsilon > 0$ such that $T + H(\lambda, \cdot)$ is regular for every $\lambda \in (-\varepsilon, \varepsilon)$. In particular, for every $\lambda \in (-\varepsilon, \varepsilon)$ the inclusion $Tx + H(\lambda, x) \ni 0$ has a solution $x = x_\lambda \in D(T) \cap G$.*

Proof. By Theorem 5, it suffices to show that there exists $\varepsilon > 0$ such that $Tx + H(\lambda, x) \not\ni 0$ for every $(\lambda, x) \in (-\varepsilon, \varepsilon) \times D(T) \cap \partial G$. To this end, assume that this is not true. Then, for some sequence $(\lambda_n, x_n) \in [-1,1] \times D(T) \cap \partial G$, we have $\lambda_n \to 0$ and $Tx_n + H(\lambda_n, x_n) \ni 0$. Since $\{(\lambda_n, x_n)\}$ is bounded and H is compact, we may assume that, for some sequence $v_n \in Tx_n$, we have $v_n \to v \in X$. Then $x_n \to x_0 = T^{-1}v \in D(T) \cap \partial G$. It follows that $Tx_0 + H(0, x_0) \ni 0$, i.e., $Tx_0 \ni 0$. Since T is L-expansive and $0 \in T(D(T) \cap G)$, we have a contradiction. This completes the proof. □

It would be interesting to see extensions of this theory to problems where the operator T is a locally defined continuous or demicontinuous operator. The invariance of domain results of Deimling [6] and Kartsatos [15] would be useful in this direction. All the results above for m-accretive operators have analogues for maximal monotone operators $T : X \supset D(T) \to 2^{X^*}$, where X is now a locally uniformly convex reflexive Banach space with locally uniformly convex dual space X^*. For results in this setting, we cite the papers [7] and [13–14]. In particular, the results of [14] contain as special cases some results of Kartsatos in [18] involving ranges of sums for perturbations of m-accretive operators.

References

[1] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1975.

[2] F. Browder, *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces*, Proc. Sympos. Pure Math., vol. 18, Part 2, Amer. Math. Soc., Providence, 1976.

[3] S. S. Chang, *Fixed Point Theory and its Applications*, Chong Qin, 1985. (Chinese)

[4] Y. Z. Chen, *The generalized degree for compact perturbations of m-accretive operators and applications*, Nonlinear Anal. **13** (1989), 393–403.

[5] I. Cioră̆nescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Acad. Publ., Boston, 1990.

[6] K. Deimling, *Zeros of accretive operators*, Manuscripta Math. **13** (1974), 365–374.

[7] Z. Ding and A. G. Kartsatos, *Nonzero solutions of nonlinear equations involving compact perturbations of accretive operators in Banach spaces*, Nonlinear Anal. (to appear).

[8] J. Dugundji and A. Granas, *Fixed Point Theory*, Polish Sci. Publ., Warszawa, 1982.

[9] M. Furi, M. Martelli and A. Vignoli, *On the solvability of nonlinear operator equations in normed spaces*, Ann. Mat. Pura Appl. **124** (1980), 321–343.

[10] M. Furi and M. P. Pera, *An elementary approach to boundary value problems at resonance*, Nonlinear Anal. **4** (1980), 1081–1089.

[11] L. Górniewicz and Z. Kucharski, *Coincidence of k-set contraction pairs*, J. Math. Anal. Appl. **107** (1985), 1-15.

[12] A. Granas, *The theory of compact vector fields and some applications to the theory of functional spaces*, Rozprawy Mat. **30** (1962).

[13] Z. Guan and A. G. Kartsatos, *Solvability of nonlinear equations with coercivity generated by compact perturbations of m-accretive operators in Banach spaces*, Houston J. Math. **21** (1995), 149–188.

[14] ———, *Ranges of perturbed maximal monotone and m-accretive operators in Banach spaces*, Trans. Amer. Math. Soc. (to appear).

[15] ———, *On the eigenvalue problem for perturbations of nonlinear accretive and monotone operators in Banach spaces*, Nonlinear Anal. (to appear).

[16] A. G. Kartsatos, *Zeros of demicontinuous accretive operators in reflexive Banach spaces*, J. Integral Equations **8** (1985), 175–184.

[17] ———, *On compact perturbations and compact resolvents of nonlinear m-accretive operators in Banach spaces*, Proc. Amer. Math. Soc. **119** (1993), 1189–1199.

[18] ———, *Sets in the ranges of sums for perturbations of nonlinear m-accretive operators in Banach spaces*, Proc. Amer. Math. Soc. **123** (1995), 145–156.

[19] ———, *Recent results involving compact perturbations and compact resolvents of accretive operators in Banach spaces*, Proceedings of the First World Congress of Nonlinear Analysts, Tampa, Florida, 1992, Walter de Gruyter, New York (to appear).

[20] V. Lakshmikantham and S. Leela, *Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press, Oxford, 1981.

[21] M. MARTELLI, *Positive eigenvectors of wedge maps*, Ann. Mat. Pura Appl. **145** (1986), 1–32.

[22] I. MASSABÒ AND C. A. STUART, *Positive eigenvectors of k-set contractions*, Nonlinear Anal. **3** (1979), 35–44.

[23] M. P. PERA, *Sulla risollubilità di equazioni nonlineari in spazi di Banach ordinati*, Boll. Un. Mat. Ital. **17-B** (1980), 1063–1074.

[24] W. V. PETRYSHYN, *Bifurcation and asymptotic bifurcation for equations involving A-proper mappings with applications to differential equations*, J. Differential Equations **28** (1978), 124–154.

Manuscript received September 1, 1994

ZOUHUA DING AND ATHANASSIOS G. KARTSATOS
Department of Mathematics
University of South Florida
Tampa, Florida 33620-5700, USA

E-mail address: ding@chuma.usf.edu, hermes@gauss.math.usf.edu