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EQUILIBRIUM VALUE AND MEASURE OF SYSTEMS OF FUNCTIONS

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This is dedicated to Professor Ky Fan

Let X be a compact Hausdorff space, and $\{f_i\}$ and $\{g_i\}$ two families of real-valued continuous functions defined on X and indexed by the same set I. We consider the question of existence of a probability Radon measure μ on X and a real number λ such that

$$\int_X g_i\,d\mu = \lambda \int_Y f_i\,d\mu, \qquad i \in I. \tag{1}$$

If λ and μ satisfy (1), then λ is called an *equilibrium value* and μ an *equilibrium measure* of the systems $\{f_i\}$, $\{g_i\}$ in this order. If μ is supported by a point x_0, then x_0 is called an *equilibrium point* of the systems $\{f_i\}$, $\{g_i\}$.

Let S be the standard $(n-1)$-simplex in $\mathbb{R}^n$, i.e. S is the set of all those points $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ with all $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$. In [3] Ky Fan has shown the existence and uniqueness of an equilibrium value and an equilibrium point of systems $\{f_1, \ldots, f_n\}$, $\{g_1, \ldots, g_n\}$ of continuous functions defined on S under the following conditions:

(a) Each f_i is convex on S.
(b) Each g_i is concave and positive on S.
(c) $f_i(x) \leq 0$ for $x \in S_i := \{(x_1, \ldots, x_n) \in S : x_i = 0\}$.
(d) For each $x \in S$, there is an index i for which $f_i(x) > 0$.

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In [3] it is actually shown that the unique equilibrium value λ is positive and is given by

$$\frac{1}{\lambda} = \min_{x \in S} \max_{1 \le i \le n} \frac{f_i(x)}{g_i(x)} = \max_{x \in S} \min_{1 \le i \le n} \frac{f_i(x)}{g_i(x)}. \tag{2}$$

The approach in [3] uses the Sperner lemma and depends on S being a simplex; here we generalize this result to the case where S is replaced by an arbitrary nonempty compact convex set in a Hausdorff topological vector space and the families of functions are indexed by a compact Hausdorff space. Our assumptions are also slightly more general than those considered in [3]. It is worthwhile to point out beforehand that we use nowhere the Sperner lemma or its analogues, but rely on minimax theorems which are consequences of the Hahn–Banach Theorem. The reader is referred to [2] and [3] for connections of the existence of an equilibrium value and an equilibrium point with other problems in mathematics.

For a compact Hausdorff space X we shall denote by $P(X)$ the space of all probability Radon measures on X endowed with the w^*-topology on the dual space of $C(X)$, the space of all real continuous functions defined on X with sup-norm. Thus $P(X)$ is a compact convex set in a Hausdorff topological vector space. As a preliminary observation we have the following lemma regarding the first equality in (2):

Lemma 1. *Let X and Y be nonempty compact Hausdorff spaces and let f and g be real functions defined on $X \times Y$ with the following properties:*

(A) *Both f and g are continuous on X and Y separately.*
(B) *For each $x \in X$ there is $y \in Y$ such that $f(x,y) > 0$.*
(C) *$g(x,y) > 0$ for all $x \in X$ and all $y \in Y$.*

Then there is $\lambda > 0$ such that

$$\frac{1}{\lambda} = \min_{x \in X} \max_{y \in Y} \frac{f(x,y)}{g(x,y)}; \tag{3}$$

and this λ satisfies the following equality:

$$\min_{x \in X} \max_{\nu \in P(Y)} \int_Y (\lambda f(x,y) - g(x,y))\, d\nu(y) = 0. \tag{4}$$

Proof. Since $\max_{y \in Y} f(x,y)/g(x,y)$ is lower semicontinuous in x, it follows that $\min_{x \in X} \max_{y \in Y} f(x,y)/g(x,y)$ exists and is positive by (B) and (C). Let λ be the reciprocal of this positive number; then (3) holds.

From (3) for each $x \in X$ there is $y \in Y$ such that $\lambda f(x,y) - g(x,y) \ge 0$. But each point of Y supports a measure in $P(Y)$, hence

$$\min_{x \in X} \max_{\nu \in P(Y)} \int_Y (\lambda f(x,y) - g(x,y))\, d\nu(y) \ge 0.$$

On the other hand, there is $x_0 \in X$ such that $\lambda f(x_0, y) - g(x_0, y) \le 0$ for all $y \in Y$. Thus $\max_{\nu \in P(Y)} \int_Y (\lambda f(x_0, y) - g(x_0, y))\, d\nu(y) \le 0$ and hence

$$\min_{x \in X} \max_{\nu \in P(Y)} \int_Y (\lambda f(x, y) - g(x, y))\, d\nu(y) \le 0.$$

We have thus established (4) and the lemma is proved.

There is an interpretation of (4) in terms of von Neumann's model of expanding economics when Y is a finite set; for this we refer to [4, p. 310], [6], and [8].

We are now ready to state and prove our result which concerns systems of functions indexed by a compact set.

THEOREM 1. *Let X be a nonempty compact convex set in a Hausdorff topological vector space and Y a nonempty compact Hausdorff space and let f, g be real-valued functions defined on $X \times Y$ which are continuous on X and Y separately with $f(\cdot, y)$ convex and $g(\cdot, y)$ concave on X for each $y \in Y$. Suppose that $D = \{x \in X : g(x, y) > 0\ \forall y \in Y\}$ is nonempty and the family of sets $\{y \in Y : f(x, y) > 0\}$, $x \in D$, is a basis for the topology of Y. Then for any positive number λ which satisfies* (4) *there is $x_0 \in X$ such that*

$$g(x_0, y) = \lambda f(x_0, y) \qquad \forall y \in Y. \tag{5}$$

PROOF. For $x \in X$ and $\nu \in P(Y)$ let

$$h(x, \nu) := \int_Y (\lambda f(x, y) - g(x, y))\, d\nu(y).$$

It follows from well-known minimax theorems (see, for example, [1], [5], [7]) and (4) that there exist $x_0 \in X$ and $\nu_0 \in P(Y)$ such that

$$\min_{x \in X} \max_{\nu \in P(Y)} h(x, \nu) = \max_{\nu \in P(Y)} \min_{x \in X} h(x, \nu) = h(x_0, \nu_0) = 0. \tag{6}$$

Obviously, (6) is equivalent to the combination of the following two inequalities:

$$h(x, \nu_0) \ge 0 \qquad \forall x \in X; \tag{7}$$

$$h(x_0, \nu) \le 0 \qquad \forall \nu \in P(Y). \tag{8}$$

We claim first that if G is a nonempty open subset of Y, then $\nu_0(G) > 0$. Choose $x \in D$ such that $\emptyset \ne \{y \in Y : f(x, y) > 0\} \subset G$. For this chosen x let $H := \{y \in Y : f(x, y) > 0\}$. Then using (7) with this x we have

$$\begin{aligned} 0 &< \int_Y g(x, y)\, d\nu_0(y) \le \lambda \int_Y f(x, y)\, d\nu_0(y) \\ &\le \lambda \int_H f(x, y)\, d\nu_0(y) \le \lambda \nu_0(H) \max_{y \in Y} f(x, y), \end{aligned}$$

from which it follows that $\nu_0(G) \ge \nu_0(H) > 0$.

Since each $y \in Y$ supports a measure in $P(Y)$, we infer from (8) that

$$g(x_0, y) \geq \lambda f(x_0, y) \qquad \forall y \in Y. \tag{9}$$

Hence if we let $G := \{y \in Y : g(x_0, y) > \lambda f(x_0, y)\}$ we have to show that G is empty in order to complete the proof. Suppose that G is nonempty. Then, since G is open, $\nu_0(G) > 0$ by the claim of the preceding paragraph and hence $\int_G (\lambda f(x_0, y) - g(x_0, y))\, d\nu_0(y) < 0$. Now using (9) we have

$$0 = h(x_0, \nu_0) = \int_G (\lambda f(x_0, y) - g(x_0, y))\, d\nu_0(y) < 0,$$

a contradiction which shows that G is empty. The proof is complete.

Returning to the conditions (a)–(d) of Ky Fan mentioned at the beginning for finite systems of functions, we see that the conditions of Theorem 1 are satisfied with $Y = \{1, \ldots, n\}$. For example, for a given i between 1 and n, let $x \in S$ be such that $x_i = 1$ but $x_j = 0$, for $j \neq i$. Then $f_j(x) \leq 0$, for $j \neq i$ by (c), and hence $f_i(x) > 0$, thus $\{j : f_j(x) > 0\}_{x \in S}$ is a basis for the discrete topology of $\{1, \ldots, n\}$. Therefore Ky Fan's existence theorem for an equilibrium value and an equilibrium point for finite systems follows from Lemma 1 and Theorem 1.

When the functions f and $-g$ are not both convex on X as in Theorem 1, we have the existence of an equilibrium measure instead of an equilibrium point:

Theorem 2. *Let X and Y be arbitrary nonempty compact Hausdorff spaces and let f and g be real-valued functions defined on $X \times Y$ which are continuous on X for each $y \in Y$ and equicontinuous on Y as families indexed by $x \in X$. Suppose that $D = \{x \in X : g(x, y) > 0\ \forall y \in Y\}$ is nonempty and the family of sets $\{y \in Y : f(x, y) > 0\}$, $x \in D$, is a basis for the topology of Y. Then for any positive number λ which satisfies* (4) *there is an equilibrium measure $\mu_0 \in P(X)$ with λ as an equilibrium value for $\{f(\cdot, y)\}_{y \in Y}$ and $\{g(\cdot, y)\}_{y \in Y}$ in this order.*

Proof. Apply Theorem 1 with X replaced by $P(X)$ and with f and g replaced by F and G respectively, where

$$F(\mu, y) = \int_X f(x, y)\, d\mu(x),$$
$$G(\mu, y) = \int_X g(x, y)\, d\mu(x), \qquad \mu \in P(X),\ y \in Y.$$

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