Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 5, 1995, 177–192

NONTRIVIAL SOLUTIONS FOR SOME SEMILINEAR PROBLEMS AND APPLICATIONS TO WAVE EQUATIONS ON BALLS AND SPHERES

A. K. Ben-Naoum[1] — J. Berkovits[2]

Dedicated to Ky Fan

1. Introduction

This paper is devoted to some multiplicity results for nontrivial solutions of nonlinear wave equations on balls and on spheres. These results follow from an abstract theorem which is obtained via a continuation theorem due to Berkovits and Mustonen [5] based on degree theoretic arguments.

The framework to which the abstract theorem will be applied is the following. We consider the equation

$$Au = N(u)$$

in $L^2(\mathcal{M})$, where $\mathcal{M}$ is a measure space, A is a densely defined closed linear operator with closed range and N is the Nemytskiĭ operator generated by a Carathéodory function $(x,s) \to g(x,s)$ from $\mathcal{M} \times \mathbb{R}$ to $\mathbb{R}$.

Our basic result is obtained for the above equation when, as in [4], one simple eigenvalue of A is crossed by the function $h(x,s) := s^{-1}g(x,s)$ as s goes from $-\infty$ to ∞. But in contrast to [4], here A is not assumed to be self-adjoint.

[1]Supported by A.G.C.D. On leave from Institut de Mathématiques, Université d'Oran, Algérie.

[2]Supported by Finnish Academy. On leave from Departement of Mathematics, University of Oulu, Finland.

We shall see that under classical assumptions the above equation has at least two nontrivial solutions. The abstract framework and abstract result are presented in the next sections.

In Section 3 we consider two applications. First we consider the problem of finding radially symmetric solutions for a semilinear wave equation on balls of the form

$$\begin{aligned} u_{tt} - \Delta u - g(t,x,u) &= 0, && (t,x) \in \mathbb{R} \times B_a^n, \\ u(t,x) &= 0, && (t,x) \in \mathbb{R} \times S_a^{n-1}, \\ u(t+T,x) &= u(t,x), && (t,x) \in \mathbb{R} \times B_a^n, \end{aligned}$$

with various hypotheses on the nonlinearity g. Other results obtained by the authors [2] concerning the spectrum of the radial symmetric wave operator are needed.

As a second application, we consider the spherical wave equation

$$u_{tt} - \Delta_n u - g(t,x,u) = 0, \qquad (t,x) \in M := S^1 \times S^n.$$

We notice that in both applications the results depend upon the space dimension n.

The discussion of nonself-adjoint applications will appear elsewhere.

2. Prerequisites

Let H be a real separable Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and corresponding norm $\| \cdot \|$. The following basic definitions are needed in the sequel.

A mapping $F : H \to H$ is:

- *monotone* if $\langle F(u) - F(v), u - v\rangle \geq 0$ for all $u, v \in H$.
- *strongly monotone* if $\langle F(u) - F(v), u - v\rangle \geq \alpha ||u - v||^2$ for all $u, v \in H$, where α is some positive constant.
- *compact* if it is continuous and for any bounded sequence (u_n) in H the sequence $(F(u_n))$ has a convergent subsequence.
- *pseudomonotone* if for any sequence (u_n) in H with $u_n \rightharpoonup u$ and
  $$\limsup \langle F(u_n), u_n - u\rangle \leq 0,$$
  we have $F(u_n) \rightharpoonup F(u)$ and $\langle F(u_n), u_n\rangle \to \langle F(u), u\rangle$.
- *of class* (S_+) if for any sequence (u_n) in H with $u_n \rightharpoonup u$ satisfying
  $$\limsup \langle F(u_n), u_n - u\rangle \leq 0,$$
  it follows that $u_n \to u$.
- *bounded* if it takes any bounded set in H into a bounded set.
- *demicontinuous* if $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.

Let $F : H \to H$ be a demicontinuous and bounded map. It is easy to see that if F is monotone, then it is pseudomonotone. Similarly, if F is strongly monotone, then $F + T$ is of class (S_+) for any compact map $T : H \to H$.

Consider now a linear operator A of the following type:

(A_1) $A : H \supset \operatorname{dom} A \to H$ is a densely defined closed linear operator with closed range and with $\ker A = \ker A^*$, where A^* is the adjoint of A.

By (A_1), $\operatorname{Im} A = (\ker A)^\perp$ and the map $K = [L|_{\operatorname{dom} A \cap \operatorname{Im} A}]^{-1}$ is bounded. We shall assume that the following holds.

(A_2) The map $K : \operatorname{Im} A \to \operatorname{Im} A \cap \operatorname{dom} A$ is compact.

Let $N : H \to H$ be a (possibly nonlinear) demicontinuous bounded map. We shall study the existence of solutions of the abstract equation

$$Au - N(u) = 0.$$

If $\dim \ker A < \infty$ we can apply the coincidence degree of Mawhin [8]. If $\dim \ker A = \infty$ we assume that N is pseudomonotone and apply again the coincidence degree [7] or the degree theory constructed in [5]. The following continuation theorem is a special case of the results proved in [5].

Continuation Theorem. *Let $G \subset H$ be an open bounded convex set and $B : H \to H$ a linear continuous map such that*

$$\ker(A - B) = \{0\}.$$

Assume that $w \in (A - B)(G \cap \operatorname{dom} A)$ and

$$Au \neq (1 - t)(Bu - w) + tN(u)$$

for all $t \in\,]0, 1[$ and $u \in \partial G \cap \operatorname{dom} A$. Then the equation $Au - N(u) = 0$ has at least one solution in $\overline{G} \cap \operatorname{dom} A$ if one of the following conditions holds:

(a) $\dim \ker A < \infty$,
(b) $\dim \ker A = \infty$, *N is pseudomonotone and B is of class* (S_+).

3. The multiplicity result

Let $\mathcal{M}$ be a measure space and H a closed subspace of $L^2(\mathcal{M})$ with the usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $A : H \supset \operatorname{dom} A \to H$ be a linear operator and assume that (A_1) and (A_2) hold. Note that the kernel of A may be infinite dimensional and the linear operator A is not assumed to be self-adjoint. However, we assume that the space H can be represented as a direct sum of three invariant subspaces with special properties.

(A_3) There exist closed subspaces $H_i \subset H$, $i = 1, 2, 3$, such that $H = H_1 \oplus H_2 \oplus H_3$ and $H_i \perp H_j$ for all $i \neq j$. The operator A is completely reduced by H_1, H_2 and H_3, that is, $P_i(\operatorname{dom} A) \subset \operatorname{dom} A$ and $AP_i u = P_i A u$ for all $u \in \operatorname{dom} A$, where $P_i : H \to H_i$ is the orthogonal projection, $i = 1, 2, 3$.

(A_4) $H_2 = \operatorname{span}\{\phi_0\}$ where $\phi_0 \in L^\infty(\mathcal{M}) \cap H$ is an eigenvector of A, $\|\phi_0\| = 1$, with corresponding eigenvalue λ_0.

(A_5) There exist constants $\underline{\theta} < \lambda_0 < \overline{\theta}$ such that

$$\langle Au, u\rangle \le \underline{\theta}\|u\|^2 \qquad \text{for all } u \in \operatorname{dom} A \cap H_1,$$

and

$$\langle Au, u\rangle \ge \overline{\theta}\|u\|^2 \qquad \text{for all } u \in \operatorname{dom} A \cap H_3.$$

The assumptions (A_3) and (A_5) are made to replace the lack of self-adjointness of A. Indeed, if $A = A^*$, then by (A_2) the spectrum $\sigma(A)$ of A consists of isolated eigenvalues with finite multiplicity, except possibly $\lambda = 0$ which may have infinite multiplicity. We can write

$$\sigma(A) := \{\ldots < \underline{\lambda} < \lambda_0 < \overline{\lambda} < \ldots\}$$

and in this case $\underline{\theta} = \underline{\lambda}$ and $\overline{\theta} = \overline{\lambda}$. Conversely, if ($A_3$), ($A_4$) and ($A_5$) hold, it is easy to see that $]\underline{\theta}, \overline{\theta}[\cap \sigma(A) = \{\lambda_0\}$.

We now give an example of a linear operator satisfying (A_3), (A_4) and (A_5) which is not self-adjoint:

Example (Telegraph operator). Let A be the abstract realization in $L^2(]0,\pi[\times]0, 2\pi[)$ of $u_{tt} - u_{xx} - \beta u_t$ with periodic and Dirichlet conditions. Then it is easy to see that $A \ne A^*$ and A has a pure point spectrum $\sigma(A) = \{j^2\}_{j=1}^\infty$. If we take $\lambda_0 = 1$, then it is not hard to prove that there exist subspaces H_1 and H_3 satisfying (A_3) such that (A_5) holds with $\underline{\theta} = 0$ and $\overline{\theta} = 3$.

We shall study the existence of multiple solutions for the equation

$$Au = N(u), \qquad u \in \operatorname{dom} A,$$

where N is the Nemytskiĭ operator generated by a function $g : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ which satisfies the Carathéodory conditions, i.e., $g(x, \cdot\,)$ is continuous for a.e. $x \in \mathcal{M}$ and $g(\,\cdot\,, s)$ is measurable for all $s \in \mathbb{R}$. Moreover, we assume that $N(H) \subset H$ and the following conditions hold.

(A_6) There exist constants a and b such that $\underline{\theta} < a < \lambda_0 < b < \overline{\theta}$ and

$$a \le \frac{g(x,s)}{s} \le b \qquad \text{for all } s \ne 0, \text{ a.e. } x \in \mathcal{M}.$$

(A_7) There exist constants c, d and $\overline{a}$ such that $a < \overline{a} < \lambda_0$ and

$$a \le \frac{g(x,s)}{s} \le \overline{a} \qquad \text{for all } |s| > d,\ \text{a.e. } x \in \mathcal{M},$$
$$\lambda_0 \le \frac{g(x,s)}{s} \le b \qquad \text{for all } 0 < |s| \le c,\ \text{a.e. } x \in \mathcal{M}.$$

Clearly by (A_6), $N(0) = 0$ and $N : H \to H$ is continuous and takes bounded sets into bounded sets. In case $\ker A$ is infinite dimensional, we also assume that $\lambda_0 \ne 0$ and $\operatorname{sgn} \lambda_0 g(x,s)$ is nondecreasing in s, which implies that $\operatorname{sgn} \lambda_0 N$ is monotone.

Our result is based on the following corollary of the continuation theorem.

Corollary 3.1. *Let $G \subset H$ be an open bounded convex set such that $0 \notin \overline{G}$. Assume that* ($A_{1,\dots,6}$) *hold. Let $w \in (A - \lambda_0 I + P_2)(G \cap \operatorname{dom} A)$ be given and assume that*

$$Au - tN(u) - (1-t)(\lambda_0 u - P_2 u) \ne (1-t)w$$

for all $u \in \partial G \cap \operatorname{dom} A$, $0 < t < 1$. Then

(a) *If $\dim \ker A < \infty$, the equation $Au - N(u) = 0$ admits at least one* (*nontrivial*) *solution $u \in \overline{G} \cap \operatorname{dom} A$.*

(b) *If $\dim \ker A = \infty$, $\lambda_0 \ne 0$ and $\operatorname{sgn} \lambda_0 g(x, \cdot\,)$ is nondecreasing for a.e. $x \in \mathcal{M}$, the equation $Au - N(u) = 0$ admits at least one* (*nontrivial*) *solution $u \in \overline{G} \cap \operatorname{dom} A$.*

Proof. Define $B := \lambda_0 I - P_2$. It follows from ($A_{3,\dots,5}$) that $\ker(A - B) = \{0\}$ and the assertion of (a) is true by the continuation theorem (a). If $\dim \ker A = \infty$ and $\lambda_0 > 0$, then N is monotone and hence pseudomonotone. Since $\lambda_0 I$ is strongly monotone and P_2 is compact, the map B is of class (S_+). Then the conclusion of (b) in case $\lambda_0 > 0$ follows from the continuation theorem (b). If $\lambda_0 < 0$ we replace A by $-A$, B by $-B$ and N by $-N$ to obtain the desired result. $\square$

We choose $w := \pm\phi_0$. Note that the only solution of the equation

$$Au - \lambda_0 u + P_2 u = \pm\phi_0$$

is $u = \pm\phi_0$. We shall construct sets G_+ and G_- such that $\phi_0 \in G_+, -\phi_0 \in G_-$ and the requirements of Corollary 3.1 are met. In the sequel we assume that ($A_{3,\dots,7}$) hold. Define

$$F_t := A - tN - (1-t)(\lambda_0 I - P_2).$$

We need some suitable a priori bounds for the solution set

$$S := \{u \in \operatorname{dom} A \mid F_t(u) = (1-t)w \text{ for some } 0 \le t < 1\}.$$

Note that $0 \notin S$ but $w = \pm\phi_0 \in S$. If $u \in S$ then

$$
\begin{cases}
t\langle N(u) - \lambda_0 u, P_1 u\rangle \leq -(\lambda_0 - \underline{\theta})\|P_1 u\|^2, \\
t\langle N(u) - \lambda_0 u, P_2 u\rangle = (1-t)\langle P_2 u - w, P_2 u\rangle, \\
t\langle N(u) - \lambda_0 u, P_3 u\rangle \geq (\overline{\theta} - \lambda_0)\|P_3 u\|^2.
\end{cases} \tag{1}
$$

In order to handle the P_2-component, we consider separately the cases $\langle P_2 u - w, P_2 u\rangle \geq 0$ and $\langle P_2 u - w, P_2 u\rangle < 0$. Define

$$D^+ := \{u \in H \mid \langle P_2 u - w, P_2 u\rangle \geq 0\} = \{u \in H \mid P_2 u = \mu w,\ \mu \leq 0 \text{ or } \mu \geq 1\}$$

and

$$D^- := \{u \in H \mid \langle P_2 u - w, P_2 u\rangle < 0\} = \{u \in H \mid P_2 u = \mu w,\ 0 < \mu < 1\}.$$

Then the first lemma goes as follows.

Lemma 3.1. *There exist $R > 1$ and $0 < \rho < 1$ such that*

$$S \cap D^- \subset \{u \in H \mid \|u\| < R,\ P_2 u = \mu w,\ \mu > \rho\}.$$

Proof. Assume that $u \in S \cap D^-$ and define $\tilde{u} := P_1 u + P_2 u - P_3 u$. It follows from (1) that

$$t\langle N(u) - \lambda_0 u, \tilde{u}\rangle < -(\lambda_0 - \underline{\theta})\|P_1 u\|^2 - (\lambda_0 - \overline{\theta})\|P_3 u\|^2 \leq 0,$$

which implies that

$$t\langle N(u) - \lambda_0 u, \tilde{u}\rangle > \langle N(u) - \lambda_0 u, \tilde{u}\rangle$$

and hence

$$0 > (\lambda_0 - \underline{\theta})\|P_1 u\|^2 + (\overline{\theta} - \lambda_0)\|P_3 u\|^2 + \int_{\mathcal{M}} (g(x,u) - \lambda_0 u)\tilde{u}.$$

Using (A_6) and (A_7) we get the estimate (cf. [4])

$$\int_{\mathcal{M}} (g(x,u) - \lambda_0 u)\tilde{u} \geq -(\lambda_0 - a)\int_{\mathcal{M}_1} |P_1 u + P_2 u|^2 - (b - \lambda_0)\|P_3 u\|^2$$

where

$$\mathcal{M}_1 := \{x \in \mathcal{M} \mid u\tilde{u} > 0,\ |u(x)| > c\}.$$

Thus

$$0 > (\lambda_0 - \underline{\theta})\|P_1 u\|^2 + (\overline{\theta} - b)\|P_3 u\|^2 - (\lambda_0 - a)\int_{\mathcal{M}_1} |P_1 u + P_2 u|^2, \tag{2}$$

which also implies

$$0 > (a - \underline{\theta})\|P_1 u\|^2 + (\overline{\theta} - b)\|P_3 u\|^2 - (\lambda_0 - a)\|P_2 u\|^2. \tag{3}$$

But $u \in D^-$ and hence $\|P_2 u\| \leq 1$ and by (3) there exists $R > 1$ such that

$$\|u\| < R \qquad \text{for all } u \in D^- \cap S. \tag{4}$$

Since $\mathcal{M}_1 \subset \mathcal{M}$ and $\overline{\theta} - b > 0$ we get from (2) the estimate

$$0 > \int_{\mathcal{M}_1} \{(a-\underline{\theta})|P_1u|^2 - (\lambda_0 - a)|P_2u|^2 - 2(\lambda_0 - a)|P_1u||P_2u|\}.$$

Since $H_2 \subset L^\infty(\mathcal{M})$ we can replace $\|P_2u\|$ by $\gamma := \operatorname{ess\,sup}_{x\in\mathcal{M}} |P_2u(x)|$ to obtain

$$0 > \int_{\mathcal{M}_1} \{(a-\underline{\theta})|P_1u|^2 - (\lambda_0 - a)\gamma^2 - 2(\lambda_0 - a)|P_1u|\gamma\}.$$

Hence the integrand is negative on a subset of $\mathcal{M}$ of positive measure. However, it is easy to see that $|P_1u| > c/2 - \gamma$ on $\mathcal{M}_1$ implying that the integrand is always positive for small values of γ (see [4]). Hence there exists $\rho_0 > 0$ such that $\gamma \geq \rho_0$ and consequently a constant $0 < \rho < 1$ such that

$$\|P_2u\| > \rho \qquad \text{for all } u \in D^- \cap S,$$

which completes the proof. □

Lemma 3.2. *There exists $R_0 > 0$ such that*

$$S \cap D^+ \subset \{u \in H \mid \|u\| < R_0\}.$$

Proof. Assume that $u \in S \cap D^+$ and define $\widehat{u} := P_1u - P_2u - P_3u$. As in the proof of Lemma 3.1 we get

$$0 \geq (\lambda_0 - \underline{\theta})\|P_1u\|^2 + (\overline{\theta} - \lambda_0)\|P_3u\|^2 - (\lambda_0 - a)\int_{\mathcal{M}} (g(x,u) - \lambda_0 u)\widehat{u}.$$

Using (A_6) and (A_7) we estimate the integral above [4]:

$$\begin{aligned}\int_{\mathcal{M}} (g(x,u) - \lambda_0u)\widehat{u} &= \int_{|u|\leq d} (g(x,u) - \lambda_0u)\widehat{u} + \int_{|u|>d} (g(x,u) - \lambda_0u)\widehat{u} \\ &\geq -C_1 - C_2\|P_1u\| \\ &\quad + (\lambda_0 - \overline{a})\int_{|u|>d} |P_2u + P_3u|^2 - (\lambda_0 - a)\|P_1u\|^2\end{aligned}$$

where C_1, C_2 are constants. Hence

$$\begin{aligned}0 \geq{}& (a-\underline{\theta})\|P_1u\|^2 - C_1 - C_2\|P_1u\| + (\overline{\theta} - \lambda_0)\|P_3u\|^2 \\ &+ (\lambda_0 - \overline{a})\int_{|u|>d} |P_2u + P_3u|^2,\end{aligned}$$

which implies that $\|P_1u\|$ and $\|P_3u\|$ remain bounded, say

$$\|P_1u\| \leq R_1 \quad \text{and} \quad \|P_3u\| \leq R_1.$$

Moreover, the integral

$$\int_{|u|>d} |P_2u + P_3u|^2 \leq \text{const.}$$

Writing $\int_{|u|>d} = \int_{\mathcal{M}} - \int_{|u|\le d}$ we easily see that there exists $R_2 > 0$ such that $\|P_2 u\| \le R_2$. Thus there exists $R_0 > 0$ such that $\|u\| < R_0$ for all $u \in S \cap D^+$ and the proof is complete. □

Theorem 3.1. *Assume that* $(\mathrm{A}_{1,\ldots,7})$ *hold. Then*

(a) *If* $\dim\ker A < \infty$, *the equation* $Au - N(u) = 0$ *admits at least two nontrivial solutions with nonvanishing* P_2*-component.*
(b) *If* $\dim\ker A = \infty$, $\lambda_0 \neq 0$ *and* $\operatorname{sgn}\lambda_0 g(x,s)$ *is nondecreasing in* s, *the equation* $Au - N(u) = 0$ *admits at least two nontrivial solutions with nonvanishing* P_2*-component.*

Proof. Obviously we may assume that $R = R_0$ in the above lemmas. We define the sets

$$G_\pm := \{u \in H \mid \|u\| < R,\ P_2 u = \pm\mu\phi_0,\ \mu > \rho\}.$$

Clearly $\overline{G_+} \cap \overline{G_-} = \emptyset$ and both sets are open bounded and convex. By Lemma 3.1 and Lemma 3.2 the requirements of Corollary 3.1 are satisfied and thus there exist $u_1 \in \overline{G_+} \cap \operatorname{dom} A$ and $u_2 \in \overline{G_-} \cap \operatorname{dom} A$ which are distinct nontrivial solutions of the equation $Au = N(u)$. □

Remark 3.1. In applications we shall frequently assume that $g(x,\cdot)$ is odd. Then it is possible that $u_2 = -u_1$ and thus Theorem 3.1 implies only the existence of one pair $\{u, -u\}$ of nontrivial solutions.

4. Applications to wave equations on balls and on spheres

4.1. Wave equation on a ball. We consider the problem of radially symmetric solutions of the semilinear wave equation

$$\begin{aligned} &u_{tt} - \Delta u - g(t,x,u) = 0, && (t,x) \in \mathbb{R} \times B_a^n,\\ &u(t,x) = 0, && (t,x) \in \mathbb{R} \times S_a^{n-1},\\ &u(t+T,x) = u(t,x), && (t,x) \in \mathbb{R} \times B_a^n. \end{aligned}$$

Here, $\Delta := \sum_{i=1}^n \partial^2/\partial x_i^2$, $B_a^n := \{x \in \mathbb{R}^n \mid ||x|| < a\}$ and $S_a^{n-1} := \{x \in \mathbb{R}^n \mid ||x|| = a\}$, with $||x|| := \big(\sum_{i=1}^n x_i^2\big)^{1/2}$.

Let $T = 2\pi$ and $2a = \pi$ and assume that the nonlinearity g is radially symmetric and 2π-periodic in time. In the case of radial symmetry the above problem can be written in the form

$$\begin{aligned} &u_{tt} - u_{rr} - \frac{n-1}{r}u_r - g(t,r,u) = 0, && (t,r) \in\ ]0,2\pi[\ \times\ ]0,\pi/2[,\\ &u(t,\pi/2) = 0, && t \in\ ]0,2\pi[,\\ &u(0,r) - u(2\pi,r) = u_t(0,r) - u_t(2\pi,r) = 0, && r \in\ ]0,\pi/2[. \end{aligned} \tag{5}$$

Let H denote the space of functions $u : [0, 2\pi] \times B^n_{\pi/2} \to \mathbb{R}$ which are radially symmetric and belong to $L^2([0, 2\pi] \times B^n_{\pi/2})$. Equipped with the usual L^2-norm and inner product $(\,\cdot\,,\,\cdot\,)$, H is a Hilbert space. If $g(t, r, u) = \lambda u$ in (5) we get a linear problem and by the standard method of separation of variables we obtain the eigenvalues

$$\lambda^n_{j,k} = \left(\frac{2\alpha_{n,j}}{\pi}\right)^2 - k^2, \qquad j \in \mathbb{Z}_+,\ k \in \mathbb{Z},$$

where $\alpha_{n,j}$ is the jth positive zero of the Bessel function of the first kind $J_\nu(x)$ of order $\nu := (n-2)/2$ (see [9], [3]). The corresponding eigenfunctions are

$$\phi^n_{j,k}(t, r) \cong \begin{cases} \cos(kt) r^{(2-n)/2} J_{(n-2)/2}(2\alpha_{n,j}\, r/\pi), & k \in \mathbb{N},\ j \in \mathbb{Z}_+, \\ \sin(kt) r^{(2-n)/2} J_{(n-2)/2}(2\alpha_{n,j}\, r/\pi), & -k \in \mathbb{Z}_+,\ j \in \mathbb{Z}_+. \end{cases}$$

Each $u \in H$ can be written as the Fourier series

$$u = \sum_{j \in \mathbb{Z}_+, k \in \mathbb{Z}} u_{j,k} \phi^n_{j,k},$$

where $u_{j,k} := (u, \phi^n_{j,k})$.

We define the abstract realization A in H of the radial symmetric wave operator with the periodic-Dirichlet conditions as follows. Let

$$\operatorname{dom} A := \left\{ u \in H \;\middle|\; \sum_{j \in \mathbb{Z}_+,\, k \in \mathbb{Z}} |\lambda^n_{j,k}|^2 \, |u_{j,k}|^2 < \infty \right\},$$

and

$$A : \operatorname{dom} A \to H, \qquad u \mapsto Au := \sum_{j \in \mathbb{Z}_+,\, k \in \mathbb{Z}} \lambda^n_{j,k} u_{j,k} \phi_{j,k}.$$

Then A is a linear, densely defined, closed self-adjoint operator with closed range. Here we use the result of [9] that $\lambda = 0$ is not an accumulation point of $\sigma(A)$.

Note that $\sigma(A)$ is unbounded from above and below. Since

$$\lambda^n_{j,+k} = \lambda^n_{j,-k},$$

we see that the multiplicity $m(\lambda)$ of $\lambda \in \sigma(A)$ is odd if and only if $\lambda = \lambda_{j_0,0}$ for some $j_0 \in \mathbb{Z}_+$.

The cases $n = 1$ and $n = 3$ are special. Then the spectrum consists of the eigenvalues

$$\lambda^1_{j,k} = (2j-1)^2 - k^2, \qquad j \in \mathbb{Z}_+,\ k \in \mathbb{Z},$$

and

$$\lambda^3_{j,k} = 4j^2 - k^2, \qquad j \in \mathbb{Z}_+,\ k \in \mathbb{Z},$$

respectively, which shows that, in both cases, the eigenvalues are isolated and 0 is the only eigenvalue of infinite multiplicity.

The nonlinearity $g(t,r,u)$ is assumed to be a Carathéodory function which is 2π-periodic in time and satisfies the growth condition

$$|g(t,r,s)| \le c_0|s| + h_0(t,r), \qquad s \in \mathbb{R},\ (t,r) \in]0,2\pi[\times B^n_{\pi/2}, \tag{6}$$

where c_0 is a constant and $h_0 \in H$. Let D denote the class of radially symmetric functions $\phi \in C^\infty(\mathbb{R} \times B^n_{\pi/2}, \mathbb{R})$ which are 2π-periodic in time for each $x \in B^n_{\pi/2}$, and have compact support in $B^n_{\pi/2}$ for each $t \in \mathbb{R}$.

We say that $u \in H$ is a *weak solution* of (5) provided that

$$\int_0^{2\pi}\int_0^{\pi/2}\bigg[u\bigg(\phi_{tt} - \phi_{rr} - \frac{n-1}{r}\phi_r\bigg) - g(t,r,u)\phi\bigg] r^{n-1}\,dr\,dt = 0$$

for every $\phi \in D$.

Let us denote by $N : H \to H$ the Nemytskiĭ operator generated by g. Then $u \in H$ is a weak solution of (5) if and only if

$$Au - N(u) = 0, \qquad u \in \operatorname{dom} A.$$

From [3] and [2] we take the following results concerning $\sigma(A)$.

Lemma 4.1. *Let n be an even integer. Then $\sigma(A)$ consists of isolated eigenvalues of finite multiplicity.*

Lemma 4.2. *Let n be an odd integer. Define $q_n := (1/\pi^2)(n-1)(n-3)$. If $\lambda \in \sigma(A)$ and $\lambda \notin [-2\pi q_n, -q_n]$, then the multiplicity of λ is finite and λ is isolated.*

Remark 4.1. If n is odd, then by the proof of Lemma 1 of [2] one can see that there is an accumulation point of the spectrum or an eigenvalue of infinite multiplicity in the interval $[-2\pi q_n, -q_n]$. In particular, in case $n = 1$ or $n = 3$, $\ker A$ is infinite dimensional and if $n > 3$, then the inverse of A is not compact.

In order to avoid the difficulties caused by the spectrum if n is odd, $n > 3$, and in order to obtain eigenvalues with multiplicity one, we shall use the following closed invariant subspaces:

$$\begin{aligned} W &:= \{u \in H \mid u(2\pi - t, r) = u(t,r) \text{ for a.e. } (t,r) \in]0,2\pi[\times]0,\pi/2[\},\\ W^\perp &:= \{u \in H \mid u(2\pi - t, r) = -u(t,r) \text{ for a.e. } (t,r) \in]0,2\pi[\times]0,\pi/2[\},\\ V &:= \{u \in H \mid u(t+\pi, r) = u(t,r) \text{ for a.e. } (t,r) \in]0,\pi[\times]0,\pi/2[\}, \end{aligned}$$

and

$$V^\perp := \{u \in H \mid u(t+\pi, r) = -u(t,r) \text{ for a.e. } (t,r) \in]0,\pi[\times]0,\pi/2[\}.$$

All the spaces above reduce the operator A. It is easy to see that

$$W := \operatorname{span}\{\phi^n_{j,k} \mid j \in \mathbb{Z}_+,\ k \in \mathbb{N}\}$$

and

$$V := \operatorname{span}\{\phi^n_{j,k} \mid j \in \mathbb{Z}_+,\ k \in \mathbb{Z},\ k \text{ even}\}$$

with the corresponding representations for $W^\perp$ and $V^\perp$.

The restriction $A|_{W\cap \operatorname{dom} A}$ is denoted by A_W and its spectrum by $\sigma(A_W)$. The multiplicity of $\lambda \in \sigma(A_W)$ is denoted by $m_W(\lambda)$. Analogous notations are used for any subspace which reduces A.

Since $\sigma(A) = \sigma(A_W) \cup \sigma(A_{W^\perp})$ and $\sigma(A) = \sigma(A_V) \cup \sigma(A_{V^\perp})$, the reduction allows us influence to the structure of the spectrum. Indeed, in case n is odd it is shown in [2] that $\sigma(A_V)$ (resp. $\sigma(A_{V^\perp})$) is made of isolated eigenvalues of finite multiplicity if $n \equiv 1 \pmod 4$ (resp. $n \equiv 3 \pmod 4$).

Naturally we must impose some symmetry and periodicity conditions for the nonlinearity g. Applying Theorem 3.1 we obtain the following theorems which distinguish different values of n and different assumptions on the nonlinearity g. Finally, we notice that in case $n = 1$ or $n = 3$ the only eigenvalues of multiplicity one are $\lambda^1_{1,0} = 1$, $\lambda^3_{1,0} = 4$, $\lambda^3_{2,0} = 16$.

THEOREM 4.1. *Let $n = 1$ or $n = 3$ and $\lambda_0 \in \sigma(A)$ with $m(\lambda_0) = 1$. Assume that the nonlinearity g satisfies conditions* (A_6) *and* (A_7) *with $\underline{\theta} := \max\{\lambda \in \sigma(A) \mid \lambda < \lambda_0\}$ and $\overline{\theta} := \min\{\lambda \in \sigma(A) \mid \lambda > \lambda_0\}$. Assume moreover that $g(t, r, \cdot\,)$ is nondecreasing. Then the problem* (5) *admits at least two nontrivial weak solutions.*

THEOREM 4.2. *Let n be an even integer, $\lambda_0 \in \sigma(A)$ and suppose* (A_6) *and* (A_7) *hold with $\underline{\theta} := \max\{\lambda \in \sigma(A) \mid \lambda < \lambda_0\}$ and $\overline{\theta} := \min\{\lambda \in \sigma(A) \mid \lambda > \lambda_0\}$.*

1) *Let $m(\lambda_0) = 1$.*

Then the problem (5) *admits at least two nontrivial weak solutions.*

2) *Let $m(\lambda_0) = 2$ and assume that $g(2\pi - t, r, s) = g(t, r, s)$, $(t, r) \in \,]0, 2\pi[\, \times \,]0, \pi/2[$, $s \in \mathbb{R}$.*

Then the problem (5) *admits at least two nontrivial weak solutions.*

THEOREM 4.3. *Let n be an odd integer and suppose g is independent of t, and $g(r, \cdot\,)$ is odd.*

1) *Let $n \equiv 1 \pmod 4$, $\lambda_0 \in \sigma(A_V)$, $m_V(\lambda_0) = 1$, and assume that* (A_6) *and* (A_7) *hold with $\underline{\theta} := \max\{\lambda^n_{j,k} \mid \lambda^n_{j,k} < \lambda_0,\ k \in \mathbb{Z},\ k$ even$\}$ and $\overline{\theta} := \min\{\lambda^n_{j,k} \mid \lambda^n_{j,k} > \lambda_0,\ k \in \mathbb{Z},\ k$ even$\}$.*

Then the problem (5) *admits at least one pair $\{-u, u\}$ of nontrivial weak solutions.*

2) *Let $n \equiv 3 \pmod 4$, $\lambda_0 \in \sigma(A_{V^\perp})$, $m_{V^\perp}(\lambda_0) = 2$ and assume that* (A_6) *and* (A_7) *hold with $\underline{\theta} := \max\{\lambda^n_{j,k} \mid \lambda^n_{j,k} < \lambda_0,\ k \in \mathbb{Z}_+,\ k$ odd$\}$ and $\overline{\theta} := \min\{\lambda^n_{j,k} \mid \lambda^n_{j,k} > \lambda_0,\ k \in \mathbb{Z}_+,\ k$ odd$\}$.*

Then the problem (5) *admits at least one pair* $\{-u, u\}$ *of nontrivial weak solutions.*

Remarks 4.2.

a) Note that if $\lambda^n_{j,k} \in \sigma(A_{V^\perp})$, then $k \neq 0$ and thus always $m_{V^\perp}(\lambda^n_{j,k}) \geq 2$.
b) In cases of the above theorems it is possible to calculate the multiplicity of a given λ_0 and the values of $\underline{\theta}$ and $\overline{\theta}$. Indeed, by the results of [2] one can see that for any n even there exists a constant C_n (which can be calculated) such that
$$|\lambda^n_{j,k}| \geq j + \frac{|k|}{2} - C_n \qquad \text{for all } j \in \mathbb{Z}_+,\ k \in \mathbb{Z}.$$
Similar estimates hold for the spectra $\sigma(A_V)$ and $\sigma(A_{V^\perp})$ in cases $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, respectively.

We now return to the above theorems and proceed to their proofs.

Proof of Theorem 4.1. It is a direct application of Theorem 3.1. □

Proof of Theorem 4.2. The first case is clear as a direct application of Theorem 3.1. But in the second one, it is easy to see that a solution of the equation
$$A_W u - N_W(u) = 0, \qquad u \in \operatorname{dom} A \cap W,$$
where $N_W = N_{|W}$, will be a weak solution of problem (5). The assumption $m_W(\lambda_0) = 2$ implies that
$$\lambda_0 = \lambda_{j_0, \pm k_0}$$
for some $j_0 \in \mathbb{Z}_+$, $k_0 \in \mathbb{Z}_+$. Hence
$$\lambda_0 \in \sigma(A_W) = \{\lambda^n_{j,k} \mid j \in \mathbb{Z}_+,\ k \in \mathbb{N}\}$$
and
$$m_W(\lambda_0) = 1.$$
Clearly assumptions $(\mathrm{A}_{1,\dots,7})$ are now satisfied for A_W and N_W with
$$H_1 = \operatorname{span}\{\phi^n_{j,k} \mid \lambda^n_{j,k} < \lambda_0,\ j \in \mathbb{Z}_+,\ k \in \mathbb{N}\},$$
$$H_2 = \operatorname{span}\{\phi^n_{j_0,k_0}\},$$
and
$$H_3 = \operatorname{span}\{\phi^n_{j,k} \mid \lambda^n_{j,k} > \lambda_0,\ j \in \mathbb{Z}_+,\ k \in \mathbb{N}\}.$$
Hence by Theorem 3.1 the proof is complete. □

Proof of Theorem 4.3. In the first case, it suffices to consider the equation
$$A_V u - N_V(u) = 0, \qquad u \in \operatorname{dom} A \cap V,$$
and to follow the proof of Theorem 4.2.

For the second case, we consider the equation

$$A_M u - N_M(u) = 0, \qquad u \in \operatorname{dom} A \cap M,$$

where $M := W \cap V^{\perp}$, that is,

$$M = \operatorname{span}\{\phi^n_{j,k} \mid j \in \mathbb{Z}_+,\ k \in \mathbb{Z}_+,\ k \text{ odd}\}.$$

The assumption $m_{V^{\perp}}(\lambda_0) = 2$ implies that

$$\lambda_0 = \lambda_{j_0, \pm k_0}$$

for some $j_0 \in \mathbb{Z}_+$, $k_0 \in \mathbb{Z}_+$ k_0 odd. Hence

$$\lambda_0 \in \sigma(A_M) = \{\lambda^n_{j,k} \mid j \in \mathbb{Z}_+,\ k \in \mathbb{Z}_+,\ k \text{ odd}\}$$

and

$$m_M(\lambda_0) = 1.$$

Clearly assumptions ($\mathrm{A}_{1,\dots,7}$) are now satisfied for A_M and N_M with

$$H_1 = \operatorname{span}\{\phi^n_{j,k} \mid \lambda^n_{j,k} < \lambda_0,\ j \in \mathbb{Z}_+,\ k \in \mathbb{Z}_+,\ k \text{ odd}\},$$
$$H_2 = \operatorname{span}\{\phi^n_{j_0,k_0}\},$$

and

$$H_3 = \operatorname{span}\{\phi^n_{j,k} \mid \lambda^n_{j,k} > \lambda_0,\ j \in \mathbb{Z}_+,\ k \in \mathbb{Z}_+,\ k \text{ odd}\}.$$

Hence by Theorem 3.1 the proof is complete. □

We close this section by a result which shows that if g is independent of t and odd, then in case n is even we obtain nontrivial solutions by crossing any eigenvalue λ_0.

THEOREM 4.4. *Let n be an even integer and $\lambda_0 \in \sigma(A)$. Assume that* (A_6) *and* (A_7) *hold with $\underline{\theta} = \max\{\lambda \in \sigma(A) \mid \lambda < \lambda_0\}$ and $\overline{\theta} = \min\{\lambda \in \sigma(A) \mid \lambda > \lambda_0\}$. Moreover, assume that g is independent of t and $g(r, \cdot\,)$ is odd. Then there exists at least one pair $\{-u, u\}$ of nontrivial weak solutions for problem* (5).

PROOF. If $m(\lambda_0) > 1$, then we can write $\lambda_0 = \lambda^n_{j_l,k_l}$, $l = 1, \dots, m$, where $0 < j_1 < j_2 < \dots < j_m$ and $0 \le |k_1| < |k_2| < \dots < |k_m|$. Define the reduction space

$$\begin{aligned} E &:= \{u \in H \mid u(t + 2\pi/|k_m|, r) = u(t, r)\} \cap W^{\perp} \\ &= \operatorname{span}\{\phi^n_{j,k} \mid j \in \mathbb{Z}_+,\ -k/|k_m| \in \mathbb{Z}_+\}. \end{aligned}$$

Clearly the multiplicity of λ_0 as an eigenvalue of A_E is one and since $N(E) \subset E$ the proof is complete. □

Remark 4.3. Note that for any given $\lambda_0 = \lambda^n_{j_l,k_l}$, $l = 1,\ldots,m$, it is sufficient to take in the previous theorem

$$\underline{\theta} = \max\{\lambda^n_{j,k} \mid \lambda^n_{j,k} < \lambda_0,\ j \in \mathbb{Z}_+,\ k/|k_m| \in \mathbb{Z}_+\},$$

$$\overline{\theta} = \min\{\lambda^n_{j,k} \mid \lambda^n_{j,k} > \lambda_0,\ j \in \mathbb{Z}_+,\ k/|k_m| \in \mathbb{Z}_+\}.$$

Similar results to those above hold for any $\lambda_0 \in \sigma(A_V)$ if $n \equiv 1 \pmod 4$ and for any $\lambda_0 \in \sigma(A_{V^\perp})$ if $n \equiv 3 \pmod 4$.

4.2. Spherical wave equation. In this section we consider briefly the equation

$$u_{tt} - \Delta_n u - g(t,x,u) = 0, \qquad (t,x) \in M := S^1 \times S^n, \tag{8}$$

where $S^n \subset \mathbb{R}^{n+1}$ is the n-dimensional sphere, Δ_n is the Laplace–Beltrami operator on the compact Riemannian manifold S^n and $g : M \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the usual growth condition.

We are looking for weak solutions u of (8), i.e. for $u \in L^2(M,\mathbb{R})$ satisfying

$$\int_M u(\phi_{tt} - \Delta_n \phi) - g(t,x,u)\phi = 0$$

for every $\phi \in C^2(M,\mathbb{R})$.

It is well known [1] that the spherical wave operator $\partial^2/\partial t^2 - \Delta_n$ is symmetric, with domain $C^2(M,\mathbb{R})$ and such that, if

$$u(t,x) \cong \sum_{j,l,m} u_{j,l,m} Y_{l,m}(x) e^{ijt}, \qquad i^2 = -1,$$

then

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_n\right) u(t,x) = \sum_{j,l,m} [l(l+n-1) - j^2] u_{j,l,m} Y_{l,m} e^{ijt},$$

where $Y_{l,m}(x)$ are spherical harmonic functions of degree l, $l = 0,1,2,\ldots$; $m = 1,2,\ldots,N(l,n)$, where

$$N(l,n) := \frac{(2l+n-1)(l+n-1)!}{(l+n-1)l!(n-1)!}, \qquad N(0,1) = 1.$$

Then $\partial^2/\partial t^2 - \Delta_n$ can be extended to be a self-adjoint operator A with domain

$$\operatorname{dom} A = \left\{ u \in L^2(M,\mathbb{R}) : \sum_{j,l,m} [l(l+n-1) - j^2]^2 |u_{j,l,m}|^2 < +\infty \right\},$$

and spectrum

$$\sigma(A) = \{ l(l+n-1) - j^2 \mid (l,j) \in \mathbb{N} \times \mathbb{Z} \}.$$

It is well known that any eigenvalue of A has finite multiplicity except $\lambda = -((n-1)/2)^2$ if n is odd (see [6]). The multiplicity of a given $\lambda \in \sigma(A)$ is $m(\lambda) := \dim \ker(A - \lambda I)$, where

$$\ker(A-\lambda I) = \operatorname{span}\{Y_{l,m}(x)e^{ijt} \mid \lambda_{l,j} = \lambda,\ m = 0,\dots,N(l,n)\}.$$

It is easy to see that $N(0,n) = 1$ for any $n \in \mathbb{Z}_+$ but $N(l,n) > 1$ otherwise. Since $\lambda_{l,+j} = \lambda_{l,-j}$, the only eigenvalue which may have multiplicity one is $\lambda^n_{0,0} = 0$. Indeed, by routine calculations one can prove that $m(\lambda^n_{l,j}) = 1$ only for the eigenvalues $\lambda^2_{0,0}, \lambda^3_{0,0}$, and $\lambda^5_{0,0}$. Hence we obtain the following results as a direct application of Theorem 3.1.

Theorem 4.5. *Let $n = 2$ and $\lambda_0 = 0$. Assume that* (A_6) *and* (A_7) *hold with $\underline{\theta} = -1$ and $\overline{\theta} = 1$. Then equation* (8) *admits at least two nontrivial weak solutions.*

Theorem 4.6.

(a) *Let $n = 3$ and $\lambda_0 = 0$. Assume that $g(t,x,s) + s$ is nondecreasing in s and* (A_6) *and* (A_7) *hold with $\underline{\theta} = -1$ and $\overline{\theta} = 2$.*

Then equation (8) *admits at least two nontrivial weak solutions.*

(b) *Let $n = 5$ and $\lambda_0 = 0$. Assume that $g(t,x,s) + 4s$ is nondecreasing in s and* (A_6) *and* (A_7) *hold with $\underline{\theta} = -1$ and $\overline{\theta} = 1$.*

Then equation (8) *admits at least two nontrivial weak solutions.*

Remark 4.4. In the proof of Theorem 4.6 we have replaced the operator A by $A + ((n-1)/2)^2 I$ which has infinite dimensional kernel. For this reason we have also assumed that $g(t,x,s) + ((n-1)/2)^2 s$ is nondecreasing.

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Manuscript received January 7, 1994

A. K. Ben-Naoum and J. Berkovits
Institut Mathématique
Université Catholique de Louvain
B-1348 Louvain-la-Neuve, BELGIQUE