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ON POSITIVE SOLUTIONS OF SOME SINGULARLY PERTURBED PROBLEMS WHERE THE NONLINEARITY CHANGES SIGN

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Dedicated to Ky Fan

In this paper, we continue our work on the problem of positive solutions of

$$
\begin{aligned}
-\varepsilon\Delta u &= g(u) \qquad \text{in } D,\\
u &= 0 \qquad \text{on } \partial D.
\end{aligned} \tag{1}
$$

Here D is a bounded domain in $\mathbb{R}^n$. We are interested in the asymptotic behaviour of positive solutions and the number of positive solutions for small positive ε in the case where $g(0) \geq 0$ but g changes sign on $[0,\infty)$. In many cases, we find the exact number of positive solutions for small ε. In particular, we improve considerably the results in [12]. Note that these results are for a restricted class of rather symmetric domains and that many of our results are new even for a ball.

In particular, we allow rather more general nonlinearities than those in [12]. (We remove the condition that $g'(0) < 0$, considerably weaken a technical condition and allow g to change sign several times.) We give a counterexample showing that the results in [12] are not true for dumbbell shaped domains. This requires a rather delicate analysis. In addition, we briefly study the case of annuli where there are noticeable differences. Note that the case $g'(0) = 0$ is much more difficult because of essential spectrum difficulties on all of $\mathbb{R}^n$.

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The more general nonlinearities we now cover include many of the nonlinearities studied by Hess [24] and Clement and Sweers [6]. They occur in many places. For example, they occur as singular limit problems for diffusion problems of competing species type as in [13], §2 (and in other population problems). Other examples appear in [32] (for different boundary conditions).

In §1, we remove the condition that $g'(0) < 0$, in §2, we allow g to have several sign changes while in §3 we discuss our counterexample. In §3 we also very briefly discuss mountain pass methods, point singularities and the case of the annulus.

1. Removal of the condition that $g'(0) < 0$

Here in this section, we remove two conditions in [12]. We use in an essential way a result in [29]. The reader should have a copy of [12] available, since we refer to it continually.

Assume that $g : \mathbb{R} \to \mathbb{R}$ is C^1, $g(0) = 0$, $g(y) < 0$ on $(0, a)$, $g(y) > 0$ on (a, ∞), there is a $\delta > 0$ such that $g'(y) < 0$ on $(0, \delta)$, and $g(y) \sim y^p$ as $y \to \infty$ where $1 < p < (n+2)(n-2)^{-1}$ if $n > 2$ and $p > 1$ if $n = 1, 2$. Finally, if $n \geq 4$ assume that there exists $\tau \leq (n-1)/(n-3)$ and $K_1 > 0$ such that either $g(y) \geq K_1(y-a)^\tau$ for $y \geq a$ and y near a or that both g is increasing on $[a, \infty)$ and that $(y-a)^{-(n+1)/(n-3)} g(y)$ is decreasing on (a, ∞).

We consider a domain $D \subseteq \mathbb{R}^n$ such that $0 \in D$, D has C^3 boundary, D is invariant under the n reflections in the coordinate planes and such that in addition, if $1 \leq i \leq n$ and if $0 < t < s < \widetilde{t}_i$, then $(I - P_i)D_{i,t} \supseteq (I - P_i)D_{i,s}$. Here P_i is the orthogonal projection onto $\operatorname{span} e_i, D_{i,s} = \{x \in D : x_i = s\}$, $\widetilde{t}_i = \sup\{x_i : x \in D\}$, and $\{e_i\}$ denotes the usual basis for $\mathbb{R}^n$. We say that such a domain is of *type* R_n.

We need to discuss the conditions on g. The major improvement is to greatly weaken the condition in [12] that $g'(0) < 0$. The last condition on g (that is, the one with 2 alternatives) is also a considerable weakening of the one in [12] (and indeed always applies in the "physical" dimensions). We suspect that it can be completely removed. Note these last conditions on g are only used (by results in [5], [21] and [36]) to ensure that the equation $-\Delta u = g(u)$ has no bounded solution on $\mathbb{R}^{n-1}$ with $u \geq a$ on $\mathbb{R}^{n-1}$). Thus we could replace the last condition on g by this condition. There are examples showing that it is a true weakening. Lastly, as we commented in [12], our results below also hold if $p = 1$ (though the proofs need some modification). We will consider the case $0 < p < 1$ in the next section.

We establish two main results.

PROPOSITION 1. *Assume that the above conditions on g and D hold and u_i are positive solutions of* (1) *for $\varepsilon = \varepsilon_i$ where $\varepsilon_i \to 0$ as $i \to \infty$. By choosing a* *subsequence if necessary there is a positive radial solution v of*

$$(2)\qquad -\Delta u = g(u)$$

on $\mathbb{R}^n$ such that $v(x) \to 0$ as $\|x\| \to \infty$ and $u_i - v(\varepsilon_i^{-1/2}x)$ converges uniformly to zero on D as $i \to \infty$.

Theorem 1. *Assume that the above conditions on D hold and that $g(y) = -y^q + Cy^p$ where $1 \le q < p < (n+2)/(n-2)$ ($p < \infty$ if $n = 1, 2$) and $C > 0$. Then* (1) *has a unique positive solution for small positive ε.*

Remarks on Theorem 1. In fact, it suffices to assume that g satisfies the conditions of Kwong and Liqun Zhang [28] if $n > 3$ or more generally that (2) has a unique positive radial solution v with the property that $v(r) \to 0$ as $r \to \infty$ and this solution is weakly non-degenerate in the sense that the linearized equation

$$-\Delta h = g'(v)h$$

has no bounded radial solution $R(r)$ on $\mathbb{R}^n$ such that $R(r) \to 0$ as $r \to \infty$ if $n \ge 3$. Analogous results to those in [28] hold for $n = 2$ by modifying the proofs in [28] or by using [31]. If $n = 1$, one can simply obtain uniqueness by using the first integral without the additional assumption and the weak non-degeneracy follows easily by an argument below (again without the extra assumption). Note that we have to be a little careful about the definition of weak non-degeneracy because, if $g'(0) = 0$, we are in the essential spectrum. (In fact, as we well see below in the proof, it suffices to assume that there are no such solutions R with exactly one positive zero.)

Proof of Proposition 1. Much of this is the same as the proof of Proposition 1 in [12]. Lemma 1 there is unchanged. (Here we obtain upper and lower bounds independent of ε_i for $\|u_i\|_\infty$.) The statement of Lemma 2 there is unchanged but the proof needs considerable modification. We commence by reminding the reader of the statement of Lemma 2 in [12]. We need to prove that there exist $\ell \in (0, a)$ and $k_2 > 0$ such that $u_i(x) \le \ell$ if $x \in D$ and $|x_j| \ge k_2\varepsilon_i^{1/2}$ for all j and all large i. To prove this, we suppose the result is false. As in the proof of Lemma 2 in [12], we can deduce that a subsequence of the u_i (rescaled) converges uniformly on compact sets to w where $w(0) > a$, $w \ge a$ on $\mathbb{R}^n$, w is bounded and $-\Delta w = g(w)$ on $\mathbb{R}^n$. Moreover, w will inherit the decreasing and evenness properties of u_i. In particular, w is even in x_j and is decreasing in x_j for $x_j \ge 0$. Hence by standard arguments (cp. the argument on pp. 7–8 in [7]) we see that $\widetilde{w} = \lim_{x_1 \to \infty} w(x)$ is a bounded solution of $-\Delta v = g(v)$ on $\mathbb{R}^{n-1}$ such that $\widetilde{w} \ge a$ on $\mathbb{R}^{n-1}$. By our assumptions and by our remarks at the beginning of this section on our assumptions on g, we see that $\widetilde{w} = a$. Thus $w(x) \to a$ as $x_1 \to \infty$. Similarly, $w(x) \to a$ as $x_i \to \pm\infty$. Since $w \ge a$, we see by the various decreasing properties of w that $w \to a$ as $\|x\| \to \infty$. This ensures that if $\ell_1 > a$, then there exists $k_2 > 0$ such that $u_i(x) \le \ell_1$ if $x \in D$ and $\varepsilon_i^{-1/2}\|x\| \ge k_2$ and if i is large. (Because, if $a < \ell_2 < \ell_1$, we can choose k_2 such that $w(x) \le \ell_2$ if $\|x\| = k_2$. Since $u_i(\varepsilon_i^{-1/2}x)$ converges to $w(x)$ uniformly on compact sets, it follows that $u_i(\varepsilon_i^{-1/2}x) \le \ell_1$ if $\|x\| = k_2$. By the decreasing properties of the u_i, it follows that $u_i(x) \le \ell_1$ if $x \in D, \|x\| \ge \varepsilon_i^{1/2}k_2$). Similarly, since $w > a$ on $\mathbb{R}^n$, we see that, if $k_3 > 0$, then $u_i(x) > a$ when $\|x\| \le \varepsilon_i^{1/2}k_3$ and i is large. Now choose $\alpha_i > 0$ such that $u_i(\alpha_i e_1) = \frac{1}{2}a$. We do a blow up argument again. We use a change of variable $X_j = \varepsilon_i^{-1/2}x_j$ for $2 \le j \le n$, $X_1 = \varepsilon_i^{-1/2}(x_1 - \alpha_i)$. By a standard argument, a subsequence of u_i will converge to a non-negative solution z of $-\Delta u = g(u)$ on $\mathbb{R}^n$ (or a half space T) such that $z(0) = \frac{1}{2}a$, z is decreasing in x_1, even in x_j for $2 \le j \le n$ and decreasing in x_j for $x_j \ge 0$, $z \le a$ always and $z = 0$ on ∂T in the half space case. We need to explain some of these properties. Firstly, let μ denote the limit of the distances from zero to $\partial\Omega$ in the new variables. Then w is defined on $\widetilde{T} = \mathbb{R}^{n-1} \times (-\infty, \mu)$. Thus we are in the half or whole space case depending on whether $\mu = \infty$ or $\mu < \infty$. Our earlier estimates imply that, if $k_2 > 0$, then $\alpha_i \ge \varepsilon_i^{1/2}k_2$ for large i. Hence, if C is a compact set in the X variables with $C \subseteq \widetilde{T}$, we see that $u_i(x) \le \ell_1$ if i is large and $X \in C$ (where x is the point corresponding to X in the original variables). Thus in the limit $z(X) \le a$ on C. Hence $z \le a$. Similarly, the set of x's corresponding to X's in C lie in $x_1 \ge 0$ and hence z will be decreasing in x_1 on $\widetilde{T}$ (because u_i decreases for $x_1 \ge 0$). This completes the construction of z. We now show that no such z can exist. We consider z on the line $P = \{x_1 e_1 : -\infty < x_1 \le \mu\}$. By our various decreasing properties, it is easy to see that $\sum_{i=2}^n \partial^2 x/\partial x_i^2 \le 0$ on P. Thus, since $-\Delta z = g(z)$ on T, it follows that $-\partial^2\widetilde{z}/\partial x_1^2 \le g(\widetilde{z})$ on $(-\infty, \mu)$ where $\widetilde{z}(x_i) = z(x_1, 0, 0, \ldots, 0)$. Hence $\widetilde{z}$ is convex (since $0 \le \widetilde{z} \le a$ and thus $g(\widetilde{z}) \le 0$ on $(-\infty, \mu)$). Moreover, $\widetilde{z}$ is strictly convex near zero since $g(\widetilde{z}(0)) = g\left(\frac{1}{2}a\right) > 0$. If $\mu = \infty$, we obtain an immediate contradiction since $\widetilde{z}$ is bounded. If $0 < \mu < \infty$, we also obtain an easy contradiction if we note that $\widetilde{z}(\mu) = 0$, $\widetilde{z} \ge 0$ and $\widetilde{z}$ is bounded. Thus no such z can occur. Hence $w < a$ somewhere in $\mathbb{R}^m$. The remainder of the proof of Lemma 2 in [12] is unchanged.

The proof of Lemma 3 in [12] is unchanged.

Most of the proof of Lemma 4 in [12] is unchanged except for one part. (A minor but important chance is that we must replace the Gidas–Ni–Nirenberg Theorem [20] by the result in [29].) We only consider the full space case. (The other case is similar.) We have to consider the possibility that there is a positive bounded solution of $-\Delta u = g(u)$ on $\mathbb{R}^n$ and $0 < m < n$ such that $u > a$ somewhere, u is increasing in x_j for $1 \le j \le m$, u is even in x_j for $m < j \le n$, u is decreasing in x_j for $x_j > 0$ and $m < j \le n$, $u \le \ell < a$ when $|x_j| \ge \tau > 0$

for $m < j \leq n$, $u < a$ somewhere on the spine $x_j = 0$ for $j < m$ and lastly that $u(x) \to 0$ as $|x_{m+1}| + \ldots + |x_n| \to \infty$ uniformly in x_j for $j \leq m$. We need to prove no such u exists. The argument in [12] is still valid provided we prove that there exists $\alpha < 0$ such that the equation $-\Delta h = g'(w)h + \alpha h$ on $\mathbb{R}^{n-m}$ has a non-trivial exponentially decaying positive solution ϕ on $\mathbb{R}^{n-m}$. Here $w(x) = \lim_{x_j \to \infty, 1 \leq j \leq m} u(x)$. The argument used in [12] to prove this is still valid if we prove that $\widehat{w} = \partial w/\partial x_1 \in W^{1,2}(\mathbb{R}^{n-m})$ and

$$W(\widehat{w}) \equiv \int_{\mathbb{R}^{n-m}} \left(\frac{1}{2}|\nabla \widehat{w}|^2 - \frac{1}{2} g'(w)\widehat{w}^2 \right) = 0.$$

Now, by [29], $w = w(r)$ where r is the polar coordinate on $\mathbb{R}^{n-m}$ and $-r^{1-\widehat{n}}(d/dr)(r^{\widehat{n}-1}w'(r)) = g(w)$ on $(0, \infty)$. Here $\widehat{n} = n - m$. Since $w \to 0$ as $r \to \infty$ and $w > 0$, we see from our assumptions on g that $g(w(r)) < 0$ for large r and hence, by the differential equation, $r^{\widehat{n}-1}w'(r)$ is increasing for large r. Hence this expression has a finite non-positive limit as $r \to \infty$. (Note that since $w \geq 0$ and $w \to 0$ as $r \to \infty$, there exist arbitrarily large r for which $w'(r) < 0$.) Hence we see that $w'(r) \leq 0$ for large r and $-w'(r) \leq Kr^{1-\widehat{n}}$ for large r. Since $w \to 0$ as $r \to \infty$, it follows easily that $w' \in L^1[1, \infty)$ and $\int_0^\infty r^{\widehat{n}-1}(w'(r))^2\, dr < \infty$ if $\widehat{n} \geq 3$. This last inequality is also true if $\widehat{n} = 1, 2$. If $\widehat{n} = 1$, this is true because $w' \in L^1[1, \infty) \cap L^\infty[1, \infty)$ while if $\widehat{n} = 2$, it follows because rw' is bounded and $w' \in L^1[1, \infty)$. Now $\widehat{w} = w'(r)P(\omega)$ where ω are the angle variables and P is a spherical harmonic. Hence we see that $\int_0^\infty r^{\widehat{n}-1}(w'(r))^2\, dr < \infty$ implies that $\widehat{w} \in L^2(\mathbb{R}^{n-m})$. Now $\widehat{w}$ is a solution of $-\Delta \widehat{w} = g'(w)\widehat{w}$. Hence by integrating over the ball B_R of radius R in $\mathbb{R}^{\widehat{n}-m}$ and integrating by parts, we see that

$$\int_{B_R} (|\nabla \widehat{w}|^2 - g'(w)\widehat{w}^2) = \int_{\partial B_R} \widehat{w} \frac{\partial \widehat{w}}{\partial r}.$$

If we prove the right hand side tends to zero as $R \to \infty$, then, since $g'(w)\widehat{w}^2 \in L^1(\mathbb{R}^{\widehat{n}-m})$ (because w is bounded and $\widehat{w} \in L^2(\mathbb{R}^{\widehat{n}-m})$, it will follow that $\nabla w \in L^2(\mathbb{R}^{\widehat{n}-m})$ and $E(\widehat{w}) = 0$, as required. Now the integral on the right hand side is a constant times $R^{\widehat{n}-2}w'(R)w''(R)$ (using the form of $\widehat{w}$). Since $r^{\widehat{n}-1}w'(r)$ is bounded and $w''(r) \to 0$ as $r \to \infty$, the required result follows. (That $w''(r) \to 0$ as $r \to \infty$ follows easily from the equation satisfied by w since $w(r) \to 0$ as $r \to \infty$). This completes the proof of Lemma 4 and thus of Lemma 3.

The remainder of the proof of Proposition 1 is the same as the proof of Proposition 1 in [12] except that we replace the Gidas–Ni–Nirenberg theorem in [20] by the main result of Li and Ni [29].

Remark. Note that Proposition 1 and the mountain pass theorem imply some results on the existence of solutions of (2) on $\mathbb{R}^n$ which appear to be new.

(We use the mountain pass theorem to obtain positive solutions of (1) for $\varepsilon = \varepsilon_i$ and then use Proposition 1 to prove the existence of positive solutions of (2).)

PROOF OF THEOREM 1. Once again this follows the corresponding proof in [12] quite closely. The first change is that the main result in [28] replaces the result in [27]. (This is to prove the uniqueness of the positive radial solution u_0 of $-\Delta u = u^p - u^q$ with the property that $u_0(r) \to 0$ as $r \to \infty$.) The existence of one solution of (1) follows as in [12]. We need to prove the uniqueness. Suppose by way of contradiction that u_i are v_i are distinct positive solutions of (1) for $\varepsilon = \varepsilon_i$ where $\varepsilon_i \to 0$ as $i \to \infty$. As there, we find that $u_i(\varepsilon_i^{-1/2}x) - u_0$ converges uniformly to zero as $i \to \infty$. A similar result holds for w_i.

As in [12], we rescale by a change of variable $X = \varepsilon_i^{-1/2}x$ and let $\widetilde{u}_i$ and $\widetilde{v}_i$ denote u_i and v_i in the rescaled variables. Thus $\widetilde{u}_i$ and $\widetilde{v}_i$ are both solutions of $-\Delta u = f(u)$ on $\varepsilon_i^{-1/2}D$ with Dirichet boundary conditions (where $f(y) = y^p - y^q$). Let $z_i = (\|\widetilde{u}_i - \widetilde{v}_i\|_\infty)^{-1}(\widetilde{u}_i - \widetilde{v}_i)$. Then z_i is a solution of

$$-\Delta z = \frac{f(\widetilde{u}_i) - f(\widetilde{v}_i)}{\widetilde{u}_i - \widetilde{v}_i} z$$

on $\varepsilon_i^{-1/2}D$ such that $\|z_i\|_\infty = 1$ and z_i is even in x_j for $1 \le j \le n$. By Proposition 1, there is a $k_1 > 0$ such that $\widetilde{u}_i(x) \le \delta$ and $\widetilde{v}_i(x) \le \delta$ if $\|x\| \le k_1, x \in \varepsilon_i^{-1/2}D$ and i is large (since $\widetilde{u}_i$ and $\widetilde{v}_i$ are both close to u_0.) Thus, by our assumptions on f, $(f(\widetilde{u}_i) - f(\widetilde{v}_i))/(\widetilde{u}_i - \widetilde{v}_i) < 0$ if i is large and $\|x\| \ge k_1$. It follows easily that $|z_i|$ has its maximum in $\|x\| \le k_1$. At the end of the proof we will show that, if $n \ge 3$,

$$|z_i(x)| \le k_2\|x\|^{2-n} \tag{3}$$

where k_2 is independent of i for $\|x\| \ge k_1$. Thus, by a standard limiting argument (similar to that in [12]) a subsequence of z_i will converge uniformly on compact sets to a non-trivial solution z_0 of

$$-\Delta z = f'(u_0)z \tag{4}$$

on $\mathbb{R}^n$ such that $\|z_0\|_\infty = 1$ and z_0 is even in x_j for $1 \le j \le n$. Moreover, if $n \ge 3$, then $|z_0(x)| \le k_2\|x\|^{2-n}$. Thus, if $n \ge 3$, then $z_0(x) \to 0$ as $\|x\| \to \infty$.

We prove that no such solution of (4) exists. As in [12], it follows by using spherical harmonics that for some integer $\widetilde{\alpha}$ (where either $\widetilde{\alpha} = 0$ or $\widetilde{\alpha} \ge n-1$) there is a non-trivial bounded solution $h(r)$ of

$$\begin{aligned} -r^{1-n}\frac{d}{dr}(r^{n-1}h'(r)) + r^{-2}\widetilde{\alpha}h &= f'(u_0)h, \\ h'(0) &= 0 \qquad (\text{if } \alpha = 0), \\ h(0) &= 0 \qquad (\text{if } \alpha > 0). \end{aligned} \tag{5}$$

Moreover, if $n \geq 3$, since $z_0(x) \to 0$ as $\|x\| \to \infty$, the formula for h in [12] (in terms of spherical harmonics) implies that $h(r) \to 0$ as $r \to \infty$.

First assume that $\widetilde{\alpha} > n-1$. As in [12], we see that $v = -u_0'$ is a positive solution of (5) for $\widetilde{\alpha} = n-1$ with $v(0) = 0$. As in our proof there, if $\widetilde{h}$ is a non-trivial solution of (5) for $\widetilde{\alpha} > n-1$ with $\widetilde{h}(0) = 0$, then the Sturm comparison theorem implies that $\widetilde{h}$ has at most one positive zero. Let c be this first zero if it exists and let $c = \infty$ otherwise. We can assume $\widetilde{h}(r) > 0$ on $(0, c)$. As in [12], we easily get a contradiction if $c < \infty$. If $c = \infty$, then $\widetilde{h}(r) > 0$ on $(0, \infty)$, and as in [12] we find that $r^{n-1}(v'(r)\widetilde{h}(r) - v(r)\widetilde{h}'(r))$ has a negative limit (possibly $-\infty$) as $r \to \infty$. To obtain a contradiction we need to consider a little more carefully the behaviour of $\widetilde{h}$ and v for large r. By Lemma 4 in [28] applied to the equation satisfied by u_0, we see that $u_0(r) \leq kr^{2-n}$ for large r. (Note that this is trivial if $n = 1, 2$.) By the equation satisfied by u_0, we easily see that $r^{n-1}u_0'(r)$ has a limit (possibly $+\infty$) as $r \to \infty$. A simple integration and the estimate of the previous sentence implies that $r^{n-1}u_0'(r)$ has a finite limit ℓ (necessarily less than or equal to zero) as $r \to \infty$. Note that this and the result that $u_0(r) \to 0$ as $r \to \infty$ implies that $u_0'(r) \to 0$ as $r \to \infty$. By integrating, we see that $\ell = 0$ if $n = 1, 2$ (since $u_0(\infty) = 0$). Now u_0' satisfies a similar equation and thus by the same argument $r^{n-1}u_0''(r)$ has a finite limit γ as $r \to \infty$. Integrating, we see that $u_0'(r) \sim \gamma(2-n)^{-1}r^{2-n}$ as $r \to \infty$ if $n > 2$. This is impossible since $r^{n-1}u_0'(r) \to \ell$ as $r \to \infty$ (where ℓ is finite) unless $\gamma = 0$. Since we can use an easier argument for $n = 1$ or 2, we see that in all cases $r^{n-1}u_0''(r) \to 0$ as $r \to \infty$. Moreover, Lemma 4 in [28] (and its analogue for $n = 1, 2$) imply that $\widetilde{h}(r)$ has a finite limit as r tends to infinity. By the equation for $\widetilde{h}$ (and since $\widetilde{h}(r) > 0$), we see as above that $r^{n-1}\widetilde{h}'(r)$ has a limit (possibly $+\infty$) as r tends to infinity. If the limit is finite one easily sees that $\widetilde{h}'(r) \to 0$ as $r \to \infty$. (If $n = 1$, we should recall that $\widetilde{h}(\infty) < \infty$.) Now suppose the limit is infinite. If $n = 1, 2$, we easily get a contradiction by integration. If $n > 3$, then $\widetilde{h}(r) > 0$ and $\widetilde{h}'(r) > 0$ for large r (since the limit is infinite). This is impossible since $\widetilde{h}(r) \to 0$ as $r \to \infty$. Thus, in all cases, $\widetilde{h}'(r) \to 0$ as $r \to \infty$. We see that $r^{n-1}u_0'(r)$ and $\widetilde{h}(r)$ have finite limits as $r \to \infty$ and that both $r^{n-1}u_0''(r)$ and $\widetilde{h}'(r)$ tend to zero as $r \to \infty$. It follows easily that $r^{n-1}(v'(r)\widetilde{h}(r) - v(r)\widetilde{h}'(r)) \to 0$ as $r \to \infty$. This contradicts a result earlier in the paragraph. Thus, if $\widetilde{\alpha} > n-1$, (5) has no solution satisfying the boundary conditions at zero and infinity.

If $\widetilde{\alpha} = n-1$, we would be looking at solutions $\widetilde{h}(r)s(w)$ where s is a first degree harmonic polynomial (and these solutions are odd). Thus this type of solution cannot be a component of z_0 (because z_0 is even). Here by a component we mean a component in the spherical harmonic decomposition of z_0.

It remains to consider the case $\widetilde{\alpha} = 0$. If $n \geq 3$, Lemma 9 in [28] implies that the solution h of (5) with $h(0) = 1$ has a non-zero limit as $r \to \infty$. (We need to know that h has at most one positive zero. We will prove this in a moment.) This is impossible since we have proved that $h(r) \to 0$ as $r \to \infty$ if $n \geq 3$. Now assume that $n = 1$ or 2. If $n = 1$, there is nothing to prove since $\widetilde{\alpha} = 0$ and $\widetilde{\alpha} = n - 1$ are the same. If $n = 2$, choose $r_0 > 0$ such that $f'(u_0(r)) \leq 0$ for $r \geq r_0$. If we choose a solution z of (5) for $\widetilde{\alpha} = 0$ with $z(r_0) = 1, z'(r_0) > 0$, it follows easily from the equation that $r^{n-1}z'(r)$ is increasing for $r \geq r_0$. Thus $r^{n-1}z'(r)$ has a positive lower bound on $[r_0, \infty)$. Since $n = 2$, it is easy to see by integrating that $z(r) \to \infty$ as $r \to \infty$. Thus only the principal solution of (5) (in the sense of [23] or [27]) can be bounded as $r \to \infty$. Hence we need to prove that if w is the solution of (5) satisfying $w(0) = 1$, then w is not the principal solution of (5). This is proved by a slight variant of Lemma 9 in [28]. Note that he uses f where we use g. As there we find that w must have a positive zero. We modify the alternative proof in [28]. We let $v(r) = ru_0'(r) + \beta u_0(r)$ with β chosen as in [28] and we find as in [28] that $Lv(r) \geq 0$ for $r \geq \widetilde{\tau}$ where $w(\widetilde{\tau}) = 0$ and $w'(\widetilde{\tau}) < 0$ (since $w(0) > 0$ and $\widetilde{\tau}$ is the first positive zero of w). Here $Lv = r^{1-n}(r^{n-1}v')' + f'(u_0)v$. Since $w(r) < 0$ for $r > \widetilde{\tau}$ (because we are assuming w has at most one positive zero), it follows by a simple computation from the equations satisfied by v and w that $T(r) = r(v'(r)w(r) - v(r)w'(r))$ is decreasing for $r > \widetilde{\tau}$. At $\widetilde{\tau}$, $v(\widetilde{\tau}) < 0$ (by the argument in the alternative proof of Lemma 8 in [28]) and hence $T(\widetilde{\tau}) < 0$. Thus T has a negative limit as $r \to \infty$. This gives an obvious contradiction since $v(r) \to 0$ as $r \to \infty, w(r) < 0$ for large r, $w'(r) \to 0$ as $r \to \infty$ and $\limsup_{r\to\infty} rv'(r) \geq 0$. That $rw'(r) \to 0$ as $r \to \infty$ follows easily from similar argument to those in the case where $\alpha > n - 1$ since the differential equation satisfied by w ensures rw' is decreasing for large r (and hence $rw'(r)$ has a limit of possibly $-\infty$). To prove that $\limsup_{r\to\infty} rv'(r) \geq 0$, note that $rv'(r) = r^2u_0''(r) + (1 + \beta)ru_0'(r)$. Since $ru_0'(r) \to 0$ as $r \to \infty$ (by earlier) it suffices to find arbitrarily large values of r where $u_0''(r) \geq 0$. If not, $u_0'(r)$ is decreasing for large r. It is easy to see this is impossible since $u_0(r) > 0$ and $u_0(r) \to 0$ as $r \to \infty$.

We still have to prove that the solution h of (5) for $\widetilde{\alpha} = 0$ with $h(0) = 1$ has at most one positive zero. We know that for each small $\varepsilon > 0$, we have at least one positive solution $\widetilde{u}_\varepsilon$ of (1) for D the unit ball which is a mountain pass point in the sense of [25]. By [19], $\widetilde{u}_\varepsilon$ is a radial function. It follows (cp. [25]) that the linearization of the partial differential equation at $\widetilde{u}_\varepsilon$ has at most one negative eigenvalue. Thus this must be true in the space of radial functions. By a slight variant of the theory in Dunford and Schwartz [18, Lemma XIII.7.49], it follows that the solution of

$$-\varepsilon r^{1-n}\frac{d}{dr}(r^{n-1}h'(r)) = f'(\widetilde{u}_\varepsilon(r))h(r),$$
$$h(0) = 1, \qquad h'(0) = 0$$

has at most 1 zero in $(0,1]$. By rescaling, it follows that the solution of

$$-r^{1-n}\frac{d}{dr}(r^{n-1}h'(r)) = f'(u_\varepsilon(r))h(r),$$
$$h(0)=1, \qquad h'(0)=0$$

has at most one zero in $(0,\varepsilon^{-1/2}]$ (where u_ε is $\widetilde{u}_\varepsilon$ rescaled).

By Proposition 1, the main result in [28], and continuous dependence, it follows that the solution of

$$-r^{1-n}\frac{d}{dr}(r^{n-1}h'(r)) = f'(u_0(r))h(r),$$
$$h(0)=1, \qquad h'(0)=0$$

has at most 1 positive zero. This ensures that the hypothesis in [28] on h is satisfied.

This completes the proof except to establish the inequality (3) for $n \geq 3$. We prove this by maximum principle arguments. We consider $T_i = \{x \in \varepsilon_i^{-1/2}D : \|x\| \geq K_1\}$. On $\partial T_i, |z_i(x)| \leq K_1^{n-2}r^{2-n}$ (since $z_i(x)=0$ on $\partial(\varepsilon_i^{-1/2}D)$ and $|z_i(x)| \leq 1$). On T_i, $\Delta z_i(x) = \alpha_i(x)z_i(x)$ where $\alpha_i(x) > 0$. We prove that $z_i(x) \leq K_1^{n-2}r^{2-n}$ in T_i. (Since we could prove a similar argument for $-z_i$, this will complete the proof.) If not, $z_i(x) - K_1^{n-2}r^{2-n}$ must have a positive maximum at $\widetilde{x} \in T_i$. Thus $z_i(\widetilde{x}) > 0$. Since r^{2-n} is harmonic, it follows that $\Delta(z_i(\widetilde{x}) - K_1^{n-2}r^{2-n}) = \alpha_i(\widetilde{x})z_i(\widetilde{x}) > 0$, which is impossible at a maximum. This completes the proof.

Remarks.

1. One can prove more results on the space of solutions of the linearization of (2) at u_0 with a little more care. One can prove that the space of solutions in $C_0(\mathbb{R}^n)$ is $(n-1)$-dimensional, there is at most one bounded solution not in $C_0(\mathbb{R}^n)$ and every bounded solution is in $C_0(\mathbb{R}^n)$ if $n = 1, 2$ or if $g(y) \sim -y^q$ as $y \to 0$ where $q \leq n(n-2)^{-1}$.
2. Most of the remarks in [12] have analogues here. There is one major change when $g'(0)=0$ and (2) has more than 1 positive solution. There is a real difficulty in proving that if u_0 is a suitable positive solution of (2), then for all small $\varepsilon < 0$ there is a positive solution of (1) which is close to u_0 when it is rescaled. The difficulty comes when we are in the essential spectrum of the linearization. We discuss this in some detail below.
3. We suspect that weak non-degeneracy holds for "generic" g but this appears difficult to prove when $g'(0)=0$. It can be proved when $g'(0) < 0$.
4. As we mentioned earlier, the uniqueness and weak non-degeneracy holds much more generally when $n = 1$. As in [12], simple examples show that they do not always hold if $n > 1$.

To complete this section, we discuss briefly the case where (2) has more than one solution. Assume that the basic assumptions of Proposition 1 hold, and that each positive solution of (2) is weakly non-degenerate. (As we will see below, this implies that (2) has only finitely many positive solutions.) Then the number of positive solutions of (1) for small ε is equal to the number of positive solutions of (2). We sketch the proof of this. We do not give the proof in detail because it is quite long and tedious, especially the reduction to the ball case. We first choose a smooth deformation D_t of D into the unit ball B in $\mathbb{R}^n$ (where $D_0 = D$, $D_1 = B$) such that each D_t is of type R_n. It is then not difficult to check that Proposition 1 holds uniformly in t (because most of our argument is concerned with behaviour near the centre and because blowing up arguments will flatten ∂D_t to a half space uniformly in t). Similarly, by examining part of the proof of Theorem 1, it is not difficult (but rather tedious) to see that there exists $\varepsilon_0 > 0$ independent of t such that every positive solution of (1) on D_t for $0 < \varepsilon \leq \varepsilon_0$ is non-degenerate in the space of even functions and that, if u_0 is a positive solution of (2), then there are $\widetilde{\varepsilon}_0$, $\alpha > 0$ such that (1) has at most 1 positive solution in $Z_{0,\alpha,t} = \{u \in C(D_t) : |u_0(x) - u(\varepsilon^{-1/2}x)| \leq \alpha \text{ on } \varepsilon^{-1/2}D_t\}$ if $0 < \varepsilon \leq \widetilde{\varepsilon}_0$ and if $0 \leq t \leq 1$. (We essentially proved this earlier for each t. The only question is the uniformity in t but this is easy to check.) Let $N(t)$ denote the number of positive solutions of (1) in $Z_{0,\alpha,t}$. By the non-degeneracy and the implicit function theorem, $N(t)$ is independent of ε. Fix $\varepsilon \in (0, \widetilde{\varepsilon}_0)$. By our earlier comments, each solution in $Z_{0,\alpha,t}$ is non-degenerate and thus, by the theory of domain variation (cp. [14]), it continues to a solution in $Z_{0,\alpha,s}$ for s near t. Thus, if $N(t) = 1$, $N(s) = 1$ if s is close to t. Now a simple limit argument shows that if $s_n \to t$ and $N(s_n) = 1$ for all n, then $N(t) = 1$. Hence it follows that $N(t)$ is independent of t if $N(0) = 1$. Hence our original claim follows if we prove the result for D the unit ball. We discuss this part more carefully because we need it in §2. By our above remarks, it suffices to show that, if u_0 is a weakly non-degenerate solution of (2), then there exist arbitrarily small ε's for which $Z_{0,\alpha,0}$ is non-empty. Let $\beta = u_0(0)$. By continuous dependence results for ordinary differential equations, we easily see that it suffices to prove that there exist γ's arbitrarily close to β for which the solution of the initial value problem

$$-r^{1-n}(r^{n-1}u'(r))' = g(u(r)), \qquad u(0) = \gamma, \qquad u'(0) = 0 \tag{6}$$

has a positive zero $\tau(\gamma)$. (Note that by a simple analysis of the equation or by the Gidas–Ni–Nirenberg theorem [19] such a solution is decreasing on $(0, \tau(\gamma))$ and thus is uniformly small for r large and $r < \tau(\gamma)$. Note also that continuous dependence ensures $\tau(\gamma) \to \infty$ as $\gamma \to \beta$.) Let $h(r)$ denote the solution of

$$-r^{1-n}(r^{n-1}h'(r))' = g'(u_0(r))h(r),$$
$$h(0) = 1, \qquad h'(0) = 0.$$

By our assumption on g' and since $u_0(r) \to 0$ as $r \to \infty$, we see that there exists $\mu > 0$ such that $g'(u_0(r)) < 0$ if $r \geq \mu$. Note that it is not difficult to show that h has a positive zero by comparing it with u_0'. It follows easily from the differential equation that h as at most one zero in (μ, ∞). Hence h has a largest zero which we denote by $\widehat{\alpha}$. We consider ε small and positive if $h(r) < 0$ on $(\widehat{\alpha}, \infty)$ while we consider small negative ε if $h(r) > 0$ on $(\widehat{\alpha}, \infty)$. Now by standard results on the differentiability of the solution with respect to initial conditions we see that the solution of (6) with initial value $\beta + \widehat{\varepsilon}$ will be $u_0(r) + \widehat{\varepsilon}h(r) + o(\widehat{\varepsilon})$ for small $\widehat{\varepsilon}$ uniformly on compact sets (of r). Thus, with our above choice of the sign of $\widehat{\varepsilon}$ we see that if $K, \delta > 0$, this solution will be less than $u_0(r)$ on $[\widehat{\alpha} + \delta, K]$. We prove that this solution crosses the axis if $|\widehat{\varepsilon}|$ is small (and the sign of $\widehat{\varepsilon}$ is as above). If not, this solution $v_{\widehat{\varepsilon}}$ must cross the solution $u_0(r)$ at a point $\ell_{\widehat{\varepsilon}}$ where $\ell_{\widehat{\varepsilon}} \to \infty$ as $\widehat{\varepsilon} \to 0$. (There is also the possibility that $v_{\widehat{\varepsilon}}(r) \to 0$ as $r \to \infty$ but this possibility can also be eliminated by a slight variant of our argument below.) Note that this argument will also show that positive solutions of (2) are isolated. (One can easily deduce from this the finiteness of the number of positive solutions of (2). A more detailed similar argument appears in §2.) Now $(\|u_0 - v_{\widehat{\varepsilon}}\|'_\infty)^{-1}(u_0 - v_{\widehat{\varepsilon}})$ is a solution of

$$-\Delta w = \frac{g(u_0) - g(v_{\widehat{\varepsilon}})}{u_0 - v_{\widehat{\varepsilon}}} w, \qquad w(\ell_{\widehat{\varepsilon}}) = 0.$$

Here $\| \, \|'_\infty$ denotes the supremum on $[0, \ell_{\widehat{\varepsilon}}]$. We now can obtain a contradiction by a limit argument very similar to that in the proof of (4). This completes the proof of our claim. Note that one can give a much easier proof of $g'(0) \neq 0$. The difficulty when $g'(0) = 0$ is that we are in the essential spectrum of the natural limit problem and hence cannot easily construct solutions directly.

Let us now assume that $n = 2$ and the assumptions of the previous paragraph hold. Let

$$\mathcal{D} = \{(u, \varepsilon) : \varepsilon > 0 \text{ and } u \text{ is a positive solution of (1)}\}.$$

A minor variant of the argument in Holzmann and Kielhofer [26] shows that each component T of $\mathcal{D}$ is a non-compact 1-manifold parametrized by $u(0)$ and thus by an interval (α, β). Moreover, solutions $(u, \varepsilon) \in T$ with $u(0)$ close to α must correspond to either ε small or ε large (and only one of these cases can occur for a particular α). A similar result holds for $(u, \varepsilon) \in T$, $u(0)$ near β. Moreover, as mentioned in [12], (1) has a unique positive solution for large ε. Thus exactly one component can contain points (u, ε) with ε large. So far, we have not needed the weak non-degeneracy. Moreover, by the results of the previous paragraph, exactly one component will contain solutions (u, ε) with ε

small and $u(\varepsilon^{-1/2}x)$ uniformly close to $u_0(r)$ (where u_0 is a solution of (2)). In fact, since each component of $\mathcal{D}$ is homeomorphic to $\mathbb{R}$ (and thus has two "ends"), we see that one component of $\mathcal{D}$ will join solutions with ε large to solutions with ε small and u (rescaled) close to a solution of (2). Any other component of $\mathcal{D}$ will have "ends" two curves of solutions emanating from different solutions of (2) (emanating as in Proposition 1). Note that each of these last components must have at least one turning point and the uniqueness in the previous paragraph implies that two different components of $\mathcal{D}$ cannot have one end the same. To understand the general structure of solutions, we need to understand which solution of (2) is the limit of the component containing solutions with ε large and which solutions of (2) are "joined" by components of $\mathcal{D}$. To determine this, we argue as follows. (We only sketch the argument.) Firstly, as we noted in the previous paragraph, the behaviour of our solutions for small ε holds uniformly in t (where we use the deformation from D to a ball used earlier). Hence we can argue much as in [11] to prove that the components of $\mathcal{D}$ change continuously in t and in particular we find that which solutions of (2) are joined by a branch of solutions of (1) (and which one is joined to solutions with ε large) is independent of t. This is not difficult but rather long and tedious. Thus we can determine which are joined by studying the case where D is a ball. In the case where D *is a ball*, the uniqueness of the initial value problem for (6) ensures that if $(u,\varepsilon) \in \mathcal{D}$ (for a ball), then $u(0) \neq u_0(0)$ for any solution u_0 of (2). Hence we see that, if T is a component of $\mathcal{D}$, then $\{u(0) : (u,\varepsilon) \in \mathcal{D}\}$ must be a component of $W = (0,\infty)\backslash Z$ where $Z = \{u_0(0) : u_0 \text{ is a solution of (2)}\}$. To see which components of W occur as components of $\widehat{\mathcal{D}} = \{u(0) : (u,\varepsilon) \in \mathcal{D}\}$, we see that the unbounded component must correspond to the unbounded component of W (since the solutions u with ε large must have $u(0)$ large). Thus the component of $\mathcal{D}$ which contains elements with ε large must contain points $(\widetilde{u},\varepsilon)$ with ε small and with $\widetilde{u}$ rescaled close to the solution $\widehat{u}$ of (2) which has $\widehat{u}(0)$ maximal. Before solving the other "connection" problems, note that the uniqueness of the positive solutions for ε small close to $u_0(r)$ implies that if $\widetilde{a} \in Z$ there can be only one component T of $\mathcal{D}$ such that the closure of $\{u(0) : (u,\varepsilon) \in T\}$ contains $\widetilde{a}$. This implies that alternate components of W (for the natural order) are the images of components of $\mathcal{D}$ and each of these is the image of exactly one component of $\mathcal{D}$. This provides a complete description since we also know that the unbounded component of W is in the image. (It is also easy to see that the component of W with zero in its closure is not.) Note that this result implies that $\{u(0) : (u,\varepsilon) \in \mathcal{D} \text{ for some } \varepsilon > 0\}$ is not easy to find explicitly and that our results and those in [26] imply that (1) has no positive solution with $u(0) \in Z$ for any $\varepsilon > 0$ and any domain D of type R_n. This does not seem at all immediate when D is not a ball. We can obtain an alternative way of determining whether $\widetilde{a} \in Z$ is in the closure of a component $\widetilde{W}$ of $\{u(0) : (u, \varepsilon) \in \mathcal{D}$ for some $\varepsilon > 0\}$ which lies above $\widetilde{a}$ or below $\widetilde{a}$. By part of the argument in the previous paragraph, this depends on the solution h of

$$-r^{1-n}\frac{d}{dr}(r^{n-1}h'(r)) = g'(u_0(r))h(r),$$
$$h(0) = 1, \qquad h'(0) = 0$$

(where u_0 is the solution of (2) with initial value $\widetilde{a}$). It is proved there that $h(r) \neq 0$ for large r and that, if $h(r) > 0$ for large r, then there exist elements $(u, \varepsilon) \in \mathcal{D}$ with ε small and $u(0)$ close to but less than $\widetilde{a}$. Hence $\widetilde{a}$ is in the closure of a component of $\widehat{W}$ below $\widetilde{a}$. (As before, it suffices to consider the case of a ball.) Similarly, if $h(r) < 0$ for large r, then $\widetilde{a}$ is the closure of a component of $\widetilde{W}$ above $\widetilde{a}$. This implies that if a_1 and a_2 are adjacent elements of Z (in the obvious sense), then the number of positive zeros of their corresponding h's must differ by an odd number. This is close to the information one might expect to obtain from a degree theory (although it is not obvious there is a reasonable degree theory if $g'(0) = 0$).

2. Some more general cases

In this section, we consider similar problems with more general g's which change sign.

We consider two cases in this section. In the first case, we assume the same conditions on g as in the previous section except we replace the condition that $g > 0$ on (a, ∞) and $g(y) \sim y^p$ as $y \to \infty$ by the conditions that there exists $b > a$ such that $g(y) > 0$ on (a, b), $g(b) = 0$ and there exists $\delta_1 > 0$ such that $g'(y) \leq 0$ on $(b - \delta_1, b)$. In addition, we assume that $\int_0^b g > 0$. If this last condition fails, our problem is trivial because (1) has no non-trivial positive solution with $\|u\|_\infty \leq b$ for every smooth domain Ω and for every $\varepsilon > 0$. This follows from [6] or [17].

Theorem 2. *Assume that the above conditions hold and that the domain D is of type R_n.*

(i) *There exists a $\delta_2 > 0$ such that (1) has a unique positive solution u_ε with $u_\varepsilon(0) \in (b - \delta_2, b)$ for each small positive ε. Moreover, $u_\varepsilon \to b$ uniformly on compact subsets of D as $\varepsilon \to 0$.*

(ii) *If $\varepsilon_i \to 0$ as $i \to \infty$ and u_i are positive solutions of (1) for $\varepsilon = \varepsilon_i$ with $\|u_i\|_\infty \leq b - \delta$ for all i, then after choosing a subsequence if necessary we can find a positive radial solution u_0 of (2) such that $u_0(r) \to 0$ as $r \to \infty$ and such that $u_i(x) - u_0(\varepsilon_i^{-1/2}x)$ tends to zero uniformly on D as $i \to \infty$. Moreover, at least one solution of this type exists for all small positive ε.*

(iii) *Moreover, if* $(y-a)g'(y) < g(y)$ *on* (a,b), *and if there is a* $\rho \geq 1$ *such that* $\lim_{y\to 0^+} y^{1-\rho}g'(y)$ *exists and is negative, then* (1) *has exactly* 2 *positive solutions* u *with* $0 < \|u\|_\infty < b$ *for all small positive* ε.

We prove this in several steps. First note that (i) was proved by Sweers [35] (in fact for general smooth domains). Thus it suffices to study positive solutions with $\|u\|_\infty \leq b-\delta$. In this case, we are back in the situation of §1. We can prove most of part (ii) by simply repeating the arguments of §1. The last part of (ii) follows from (i) and a simple degree argument. Thus it remains to prove (iii). By the arguments in the proof of Theorem 1, it suffices to prove that (2) has a unique positive radial solution and that this solution is weakly non-degenerate in the sense of §1 in the space of radial functions. (Thus we really only have to study an ordinary differential equation.)

Lemma 1. *Under the conditions of Theorem* 2(iii) *on* g, *there is a unique positive radial solution* u_0 *of*

$$-r^{1-n}(r^{n-1}u'(r))' = g(u(r)) \qquad \text{on } (0,\infty)$$

such that $0 < \|u_0\|_\infty < b$ *and* $u_0(r) \to 0$ *as* $r \to \infty$. *Moreover, this solution is weakly non-degenerate in the sense of* §1 *in the space of radial functions.*

Proof. *Step 1.* We prove weak non-degeneracy first. We first note that if γ is large then

$$\gamma g(y) - yg'(y) < 0 \qquad \text{on } (0,a). \tag{7}$$

Since $g(y) < 0$ on $(0,a)$, this is obvious except for y near zero or a. Our assumptions ensure that $(u-a)^{-1}g(u)$ is strictly decreasing on (a,b) and is positive. It follows that $g'(a) > 0$. It is then easy to check that $\gamma g(y) - yg'(y) < 0$ near a if $\gamma > 0$. By our assumptions, $g'(y) \sim Cy^{\rho-1}$ for small positive y where $C < 0$. Thus, by integrating, $g(y) \sim \rho^{-1}Cy^\rho$ for small positive y. Hence our claim is true for small y (since $C < 0$). Hence (7) is true. Let $w(r)$ denote the solution of

$$\begin{gathered}-r^{1-n}(r^{n-1}w'(r))' = g'(u_0(r))w(r),\\ w(0) = 1, \qquad w'(0) = 0.\end{gathered}$$

We need to prove that w is not bounded if $n = 1,2$ and that w does not tend to zero as $r \to \infty$ if $n \geq 3$. The weak non-degeneracy follows from this. First assume that $n \geq 2$. We will also prove that w has exactly 1 zero in $(0,\infty)$. We first claim that w has no zeros in $(0,\widehat{r}]$ where $u_0(\widehat{r}) = a$. Note that it is well known and easy to prove that $u_0'(r) < 0$ on $(0,\infty)$ and $u_0(0) > a$. Our claim follows from a simple comparison argument since $u_0(r) - a$ is a positive solution of

$$-Lh(r) = (u_0(r) - a)^{-1} g(u_0(r))h(r),$$
$$h'(0) = 0,$$

and since $(u_0(r) - a)^{-1} g(u_0(r)) > g'(u_0(r))$ on $(0, \widehat{\tau})$. Here $Lh = r^{1-n}(r^{n-1}h')'$.

Hence if w has a zero, its first zero $\widetilde{\tau}$ will satisfy $\widetilde{\tau} > \widehat{\tau}$. If $\beta > 0$, let $\gamma = 1 + 2\beta^{-1}$ and $v(r) = ru_0'(r) + \beta u_0(r)$. For future reference note that β small corresponds to γ large. By p. 591 of [28], $L_1 v = \beta(u_0 g'(u_0) - \gamma g(u_0))$ where $L_1 s = Ls + g'(u_0)s$. Note that we use g where f is used in [28]. Hence we see from (7) that

$$L_1 v(r) \geq 0 \tag{8}$$

on $[\widehat{\tau}, \infty)$ if β is small. Now $L_1 w(r) = 0$ on $[\widetilde{\tau}, \infty)$ since w is a solution of the linearized equation. By a simple computation using the equation satisfied by w, one easily finds that

$$(r^{n-1}(v(r)w'(r) - w(r)v'(r)))' = -r^{n-1}(L_1 v)(r)w(r). \tag{9}$$

Now suppose that w has 2 (or more) positive zeros. Let $\widetilde{\tau} < \mu$ be the first 2. Since $w(0) > 0$, $w(r) < 0$ on $(\widetilde{\tau}, \mu)$. By what we have already proved, $\widetilde{\tau} > \widehat{\tau}$. Hence by (8) and (9), $W(r) = r^{n-1}(v(r)w'(r) - w(r)v'(r))$ is increasing on $[\widetilde{\tau}, \mu]$ if β is small. This is impossible since $W(\widetilde{\tau}) = \widetilde{\tau}^{n-1} v(\widetilde{\tau})w'(\widetilde{\tau}) > 0$ if β is small (since $w'(\widetilde{\tau}) < 0$ and $v(\widetilde{\tau})$ is close to $\widetilde{\tau} u_0'(\widetilde{\tau}) < 0$) while $W(\mu) = \mu^{n-1} v(\mu) w'(\mu) < 0$ (since $w'(\mu) > 0$). Thus w has no zero on $[\widetilde{\tau}, \infty)$ and hence $\widetilde{\tau}$ is the only zero. (We will prove in a moment that there must be such a zero.) Similarly, since $w(r) < 0$ on $(\widetilde{\tau}, \infty)$, $W(r)$ is increasing on $[\widetilde{\tau}, \infty)$, while, as before, $W(\widetilde{\tau}) > 0$ if β is small. We will prove that $\liminf_{r\to\infty} W(r) \leq 0$ if $n = 2$ and w is bounded or if $n \geq 3$ and $w(r) \to 0$ as $r \to \infty$. This gives a contradiction (and will complete the proof of weak non-degeneracy where $n > 1$ apart from proving w has a zero in $[\widehat{\tau}, \infty)$). First assume that $n \geq 3$. As in the proof of Proposition 1 in §1 (where we modified the proof of Lemma 4 in [12]), we easily see that $r^{n-1}w'(r)$ is bounded and positive and $r^{n-2}w$ is bounded. (Note that since $w(r) \to 0$, w is the principal solution in the sense of [23].) Similarly $r^{n-2}u_0'(r)$ is bounded and $u_0(r) \to 0$ as $r \to \infty$. Hence $v(r) \to 0$ as $r \to \infty$. It follows easily that $r^{n-1}w'(r)v(r) \to 0$ as $r \to \infty$. Now $rv' = r^2 u_0'' + (1+\beta) r u_0'$ by the formula for v. Now $r^2 u_0''(r) \geq K > 0$ for large r is impossible by integrating twice and using that u_0 is bounded. Since $ru_0'(r) \to 0$ as $r \to \infty$, it follows that $\liminf_{r\to\infty} rv'(r) \leq 0$. Since $r^{n-2}w(r)$ is bounded and negative, it follows that $\limsup_{r\to\infty} r^{n-1}w(r)v'(r) \leq 0$. Since $r^{n-1}w'(r)v(r) \to 0$ as $r \to \infty$, it follows that $\limsup_{r\to\infty} W(r) \leq 0$. This gives an obvious contradiction. If $n = 2$, the argument is the same except we must be a little more careful to prove that $v(r) \to 0$ as $r \to \infty$. The difficulty is to check that $ru_0'(r) \to 0$ as $r \to \infty$. As before, since $n = 2$, we can check that $ru_0'(r)$ has a limit as $r \to \infty$ (by the equation satisfied by u_0). Then the boundedness of u_0 forces $ru_0'(r)$ to tend to zero as $r \to \infty$.

It remains to prove that w must have a zero in $[\widehat{r}, \infty)$ if $n > 1$. There are several ways to prove this. Suppose not; then $w(r) > 0$ on $[\widehat{r}, \infty)$ and hence on $(0, \infty)$. By differentiating the equation for u_0, one easily finds that $-L_1 u_0' = -(n-1)r^{-2}u_0' > 0$ (since $n > 1$). In the same way as we derived (9), we see that $(r^{n-1}(u_0'(r)w'(r) - w(r)u_0''(r)))' = -r^{n-1}L_1(u_0')w(r) > 0$ by the formula for $L_1 u_0'$ and since $w(r) > 0$. Hence $\widetilde{W}(r) \equiv r^{n-1}(u_0'(r)w'(r) - w(r)u_0''(r))$ is increasing. Since $n > 1$ and w and u_0 are regular at zero, $\widetilde{W}(r) \to 0$ as $r \to 0$. Hence $\widetilde{W}(r) > 0$ for $r > 0$ and thus $-w^{-1}u_0'$ is strictly increasing. Since $w(0)^{-1}u_0'(0) = 0$, it follows that there is a $K > 0$ such that $-w(r)^{-1}u_0'(r) \geq K$ for r large, that is, $w(r) \leq -K^{-1}u_0'(r)$ for r large. Hence $w(r) \to 0$ as $r \to \infty$. Much as earlier in this step it follows that $\limsup_{r \to \infty} \widetilde{W}(r) \leq 0$. This is impossible since $\widetilde{W}(0) = 0$ and $\widetilde{W}$ is strictly increasing. Hence w has a positive zero.

If $n = 1$, then $u_0'(r)$ satisfies the same equation as w. Since u_0' can be the only solution of this equation bounded at infinity (cp. [17], p. 1552) and since u_0' does not satisfy the boundary condition at zero, the result is also proved if $n = 1$. (Here we do not use the assumption that w has a positive zero.) This completes Step 1.

Step 2. Uniqueness. If $n = 1$, the uniqueness of u_0 follows easily by using the first integral.

Now suppose $n > 1$. We will prove uniqueness indirectly by looking at solutions of

$$-Lu = g(u), \qquad u'(0) = 0, \qquad u(R) = 0$$

for R large (or equivalently the radial solutions of (1) on the unit ball B for ε small). We will prove that this equation has a unique positive solution of the type in Theorem 2(ii) for large R and deduce the uniqueness of the positive solution of (1) with norm not close to b. We will use some of the ideas in the last two paragraphs of §1 (the easy ones).

To do this, first note that (2) has only a finite number of positive solutions. To see this note that by part of the second last paragraph of §1, each positive solution of (2) is isolated and thus $Y = \{u_0(0) : u_0(r)$ is a positive solution of (2), $u_0(r) \to 0$ as $r \to \infty\}$ consists of isolated points. (Note that each such u_0 is decreasing and hence if two solutions are close on compact sets they are uniformly close on $[0, \infty)$.) Moreover, it is easy to see that Y is closed (since by [17], $u_0(0) \geq c$ where $\int_0^c g(y)\,dy = 0$ and $c > 0$) and since it is easy to show that, if $\widetilde{u}$ is a non-negative solution of (2) with $\beta - \delta \geq \widetilde{u}(0) \geq c$ which is decreasing for $r \geq 0$, then $\widetilde{u}(r) \to 0$ as $r \to \infty$. Moreover, Y is bounded above by $\beta - \delta$, since if there existed an element t of Y larger than $b - \delta$ then an easy continuous dependence argument (cp. below) would imply that there would exist solutions u of (1) on a ball for all small $\varepsilon > 0$ with $u(0)$ close to t and thus $u(0)$ larger than $b - \delta$. Moreover, by their construction these solutions are small except close to zero. This is impossible by part (i) of the theorem. Thus Y is compact and consists of isolated points and hence is finite. This proves our claim.

Next we note that, if u_0 is a positive solution of (2), then, for all small positive ε, there is a positive solution u_ε of (1) on B near u_0. (Here by near we mean $u_\varepsilon(r) - u_0(\varepsilon^{-1/2} r)$ is uniformly small on $[0, \varepsilon^{-1/2}]$). This is proved in the second last paragraph in §1. Note that we only need the results of the second last paragraph of §1 in the case of a ball.

Hence our uniqueness claim will follow if we prove that (1) has exactly two positive solutions for D a ball if ε is small. (Remember that there is a unique large solution u_ε.) It suffices to prove that any solution other than the large solution has degree -1. Here we use that the maximal solution is non-degenerate and stable and hence has index 1 and that the sum of the indices of the positive solutions is zero. The last result is proved by continuing to large ε and note that there is no positive solutions for large ε since $|g(y)| \leq K|y|$ on $[0, b]$. More precisely, here we are choosing C large (depending on ε) and considering the fixed point indices of the fixed points of the map

$$u \to (-\Delta + CI)^{-1}(\varepsilon^{-1} g(u) + Cu)$$

on the set of non-negative continuous radial functions on the unit ball.

It remains to prove that any positive solution u of (1) on B with $\|u\|_\infty \in (0, b)$ other than the large solution has index -1 if ε is small. Since we know that these solutions are non-degenerate (by Lemma 1 and by a similar argument to part of the proof of Theorem 1), we can use the well known result for the degree of a non-degenerate solution (cp. Lloyd [30], Theorem 8.1.1) and a standard argument to show that it suffices to prove that the linear eigenvalue problem

$$\begin{gathered} -Lh - \varepsilon^{-1} g'(u_\varepsilon) h = \gamma h \qquad \text{on } [0,1], \\ h'(0) = 0, \qquad h(1) = 0, \end{gathered}$$

has exactly one negative eigenvalue counting multiplicity for small positive ε. Here u_ε is the radial solution of (1) which when rescaled (to $\widetilde{u}_\varepsilon$) is close to u_0 for small ε, where u_0 is defined in Lemma 1. Equivalently, by rescaling, we could consider the problem

$$-Lh - g'(\widetilde{u}_\varepsilon) = \gamma h \qquad \text{on } [0, \varepsilon^{-1/2}],$$
$$h'(0) = 0, \qquad h(\varepsilon^{-1/2}) = 0.$$

By a well known slight strengthening of a result in Dunford and Schwartz [18, Lemma XIII.7.9], it suffices to prove that the solution h_ε of

$$-Lh = g'(\widetilde{u}_\varepsilon)h, \qquad h(0) = 1, \qquad h'(0) = 1$$

has exactly one zero in $(0, \varepsilon^{-1/2})$. (Note that, by our earlier comments, $h_\varepsilon(\varepsilon^{-1/2}) \neq 0$.) Since $\widetilde{u}_\varepsilon$ is uniformly close to u_0 we see from continuous dependence that, on compact subsets of $[0, \infty)$, h_ε is uniformly close to the solution h_0 of

$$-Lh = g'(u_0)h, \qquad h(0) = 1, \qquad h'(0) = 0$$

for ε small. We showed in the proof of Step 1 that h_0 has exactly one positive zero. Thus h_ε will have at least one positive zero t_ε which is not large and any other positive zero in $(0, \varepsilon^{-1/2})$ must be large when ε is small. This latter possibility we can eliminate by a limiting argument almost the same as in the derivation of (4) in the proof of Theorem 1 (but a little easier).

Hence h_ε has exactly one zero in $(0, \varepsilon^{-1/2})$ and our claim follows. This completes the proof of Lemma 1 and hence of Theorem 2.

Remarks.

1. The proof can be greatly simplified when $g'(0) \neq 0$. (iii) is a little surprising since we have very weak conditions on g on $(0, a)$.
2. There are analogous results if we assume that $g(y) > 0$ for $y > a$ and either (i) $g(y) \to M > 0$ and $yg'(y) \to 0$ as $y \to \infty$ or (ii) there exists $C > 0$ and $q \in (0, 1)$ such that $y^{1-q}g'(y) \to C$ as $y \to \infty$. One difference in these cases is that the maximal solution is large on most of the domain. The results are proved by combining the ideas here with those in [9] and by using the weak Harnack inequality and the method of sweeping families of subsolutions (as in Sweers' paper [35]). In particular, the last two ideas are used to prove that the set of positive solutions of (2) which tend to zero at infinity are bounded in the uniform norm in this case.
3. One can use the remarks in [12] to show that (iii) may fail without the additional assumption on g. (One modifies the nonlinearity in [12] for large y.) In the case of non-uniqueness one can prove analogues of the results of the last two paragraphs of §1. In particular, under a weak non-degeneracy assumption the number of positive solutions of (1) for small positive ε is one more than the number of positive radial solutions of (2). It seems likely that the conditions for uniqueness can be greatly weakened.

We now consider another case. These nonlinearities are ones which have 3 positive zeros. We assume that g is C^1 and that there exist a_1, a_2, a_3 such that $0 < a_1 < a_2 < a_3$, $g(y) > 0$ on $(0, a_1)$, $g(y) < 0$ on (a_1, a_2) and $g(y) > 0$ on (a_2, a_3). We are interested in positive solutions with $\|u\|_\infty \in (a_1, a_3)$. Note that positive solutions with $\|u\|_\infty \in (0, a_1)$ are quite well understood for small ε by the results in [9] and that the maximum principle implies that there are no positive solutions u with $\|u\|_\infty = a_i$ for $i = 1, 2, 3$. Note also that we must assume that $\int_{a_1}^{a_3} g(y)\,dy > 0$ since otherwise by the results in [17] there are no positive solutions of the required type. In addition, we assume that the function $y \to g(y - a_1)$ satisfies the same conditions on $[0, a_3 - a_1]$ as the g in the first part of this section (with b replaced by $a_3 - a_1$ and a by $a_2 - a_1$). Moreover, we assume that, if $n \geq 3$, there exist $q \in (1, n(n-2)^{-1}]$ and $\widetilde{C} > 0$ such that $g(y) \geq \widetilde{C} y^q$ for small positive y. (In fact, we could replace this assumption by the condition that, if $3 \leq m \leq n$, then $-\Delta u = g(u)$ has no positive solution $\widetilde{u}$ on $\mathbb{R}^m$ such that $\widetilde{u} \to 0$ as $\|x\| \to \infty$, $\widetilde{u}$ is even in x_i, $\partial\widetilde{u}/\partial x_i \leq 0$ if $x_i \geq 0$ (for $1 \leq i \leq m$) and $u(0) \in (0, a_3)$.).

We construct approximate solutions of (1) for small positive ε. Let z_0 denote the unique increasing solution of $-z''(t) = g(z(t))$, $z(0) = 0$, $z(\infty) = a_1$, let $n(x)$ denote the inward normal to ∂D at $x \in \partial\Omega$ and assume u_0 is a positive solution of $-\Delta u = g(u)$ in $\mathbb{R}^n$ such that $\|u_0\|_\infty \in (a_2, a_3)$ and $u(x) \to a_1$ as $\|x\| \to \infty$. Note that as in [9], points of D near ∂D can be uniquely written in the form $s + tn(s)$ were $s \in \partial\Omega$, $t \geq 0$ and t is small. Define

$$S_{\varepsilon,u_0}(x) = \begin{cases} u_0(\varepsilon^{-1/2}x) & \text{if } x \in D \text{ and } x \text{ is not close to } \partial D, \\ z_0(\varepsilon^{-1/2}t) & \text{if } x \text{ is close to } \partial D. \end{cases}$$

(To be completely precise we should specify exactly what we mean by close but it turns out that it does not really matter because for ε small the difference between different choices is uniformly small.)

Theorem 3. *Assume that the above conditions on g hold and that D is of type R_n. Then*

(i) *There exists $\delta > 0$ such that, if ε is small, there is a unique positive solution u_ε with $\|u_\varepsilon\|_\infty \in (a_3 - \delta, a_3)$. Moreover, u_ε converges uniformly to a_3 on compact subsets of D as $\varepsilon \to 0$.*

(ii) *Suppose that $\varepsilon_i \to 0$ as $i \to \infty$ and that u_i are positive solutions of* (i) *for $\varepsilon = \varepsilon_i$ such that $a_1 < \|u_i\|_\infty \leq a_3 - \delta$ for all i. Then we can choose a subsequence and a positive radial solution u_0 of*

$$\begin{aligned} -\Delta u &= g(u) \qquad \text{in } \mathbb{R}^n, \\ u(x) &\to a_1 \qquad \text{as } \|x\| \to \infty \end{aligned}$$

such that $u_i - S_{\varepsilon_i,u_0}$ tends to zero uniformly on D as $i \to \infty$. Moreover, at least one solution of this type exists for all small positive ε.

(iii) *If $g'(y) < (y-a_2)^{-1}g(y)$ on (a_2,a_3) and if there exists $s \geq 0$ such that $-(y-a_1)^{-s}g'(y) \to C \in (0,\infty)$ as $y \to a_1$ (for $y > a_1$), then* (1) *has exactly* 2 *positive solutions v with $a_1 < \|v\| < a_3$ for all small positive ε.*

REMARKS.

1. The proof of (ii) can be simplified a great deal if either $g(0) > 0$ or $g'(0) > 0$ by using the results in [9] and showing that for small ε there is a minimal positive solution u with $\|u\|_\infty \in (0,a_3)$.
2. We could prove variants with different behaviour for $y > a_2$ (much as in Remark 2 after Theorem 2).
3. Note that the solutions in Theorem 2(ii) with norm not close to a_3 have 2 sharp layers, a layer near $y = 0$ and a boundary layer near ∂D.

PROOF OF THEOREM 3. As before, (i) follows from [35].

(ii) The main new ingredient is to prove that if u_i is less than a_1 (and not close to a_1) at points not too close to ∂D or 0, then u_i is uniformly small except near zero. The rest of the proof is very similar to that of Theorem 2(ii).

By the theory in [9] there exists $\overline{\lambda} > 0$ such that (1) on B_1 has a positive solution ϕ for $\varepsilon = \overline{\lambda}^{-1}$ with $\|\phi\|_\infty = \mu < a_1$. (In fact, this holds for many $\overline{\lambda}$.) Now, by [9], the function

$$w(x) = \begin{cases} \phi(\overline{\lambda}^{1/2}\varepsilon^{-1/2}(x-x_0)) & \text{if } \|x-x_0\| < \varepsilon^{1/2}\overline{\lambda}^{-1/2}, \\ 0 & \text{otherwise}, \end{cases}$$

is a subsolution of (1) on D if $x_0 \in D$ and $d(x_0,\partial D) \geq \varepsilon^{1/2}\overline{\lambda}^{-1/2}$. By using the method of sweeping families of subsolutions as in [9] or [6], it follows that, if u is a solution of (i), either

$$u(x) \geq \mu \quad \text{when } x \in D,\ d(x,\partial D) \geq \varepsilon^{1/2}\overline{\lambda}^{-1/2}$$

or

$$u(x) < \mu \quad \text{at some point in the ball with centre zero and radius } \varepsilon^{1/2}\overline{\lambda}^{-1/2}.$$

We refer to these as the former and latter cases respectively. In the latter case, the various decreasing properties of u then ensure that $u(x) < \mu$ whenever $x \in D$, $d(x,S) \geq \varepsilon^{1/2}\overline{\lambda}^{-1/2}$ where S is the spine $= \{x \in D : x_i = 0 \text{ for some } i\}$. Thus in the latter case $u < \mu$ except near the spine. Moreover, in this latter case, given $\delta > 0$, there exists $\widetilde{\tau} > 0$ such that $u(x) \leq \delta$ if $d(x,S) \geq \varepsilon^{1/2}\widetilde{\tau}$ (and thus u is uniformly small except within order $\varepsilon^{1/2}$ of the spine). To prove this, it suffices to prove that if $\varepsilon^{-1/2}d(x^i,S) \to \infty$ as

$i \to \infty$ and if x^i is in a small ball in D with centre 0 and radius independent of i, then $u_i(x^i) \to 0$ as $i \to \infty$. (This follows because of the decreasing properties of u_i.) If not, we shift the origin to x^i and rescale the variables by $\varepsilon_i^{1/2}$ to obtain $\widetilde{u}_i$ with $\widetilde{u}_i(0) \geq K > 0$. Much as in [12], a subsequence of $\widetilde{u}_i$ will converge uniformly on compact sets to a non-negative bounded solution $\widehat{u}$ of $-\Delta u = g(u)$ on $\mathbb{R}^n$ with $\widehat{u}(0) \geq K$. Moreover, by our choice of x^i, if B is ball, then $\widetilde{u}_i(x) \leq \mu$ on B for i large. Thus $\widehat{u}(x) \leq \mu$ on B. Hence $\widehat{u} \leq \mu$ on $\mathbb{R}^n$. This contradicts Proposition 1 in [9]. This proves our claim.

We continue to examine this latter case. Let S' consist of the n axes. Choose $t_j > 0$ such that $u(t_j e_j) = \frac{1}{2}a_1$ (t_j certainly exists since $u(0) > a_1$). Suppose u_i are positive solutions for $\varepsilon = \varepsilon_i$ where $\varepsilon_i \to 0$ as $i \to \infty$ which are each not the large solution and let t_j^i denote the corresponding t_j. If (after choosing a subsequence if necessary) one finds that $|t_j^i| \leq K\varepsilon_i^{1/2}$ for all i and j, then we see by a standard blowing up argument at zero that we have a solution of $-\Delta u = g(u)$ in $\mathbb{R}^n$ such that $u(0) > a_1$, u is even in each x_j, $\partial u/\partial x_j \leq 0$ if $x_j \geq 0$ and $u(Ke_j) \leq \frac{1}{2}a_1$ for all j. It follows easily that $u(x) \leq \frac{1}{2}a_1$ if $\|x\|$ is large (by the decreasing properties). By a simple blowing up argument (applied to u with the origin shifted) rather like, but simpler than the one at the end of the previous paragraph, one finds that $u(x) \to 0$ as $\|x\| \to \infty$. We will show at the end of the proof of this part (part (ii)) that such a u cannot exist. Assuming this for the moment, the only possibility is that there exists j such that $\varepsilon_i^{-1/2} t_j^i \to \infty$ as $i \to \infty$. Without loss of generality, we can assume $j = 1$. By shifting the origin to $t_1^i e_1$ and by a standard blowing up argument, we see that we have a bounded positive solution of $-\Delta u = g(u)$ in either $\mathbb{R}^n$ or a half space $T = \{x \in \mathbb{R}^n : x_1 \geq -K_1\}$ such that u is increasing in x_1, even in x_j for $j \geq 2$, $\partial u/\partial x_j \leq 0$ if $x_j \geq 0$ and $j \geq 2$, $u(0) = \frac{1}{2}a_1$ and $u = 0$ on ∂T in the half space case. (Which case occurs depends on whether $\varepsilon_i^{1/2} d(t_1^i e_1, \partial D)$ is bounded or not.) Moreover, our estimate that solutions of (1) are small away from the coordinate planes S gives in the blowing up limit that, given $\varepsilon > 0$, there exists $K > 0$ such that $u(x) \leq \varepsilon$ if $x_j \geq K$ for $2 \leq j \leq n$ uniformly in x_1. In either case, consider $\lim_{x_1 \to \infty} u(x)$. As in [12], we easily see that this is a solution v on $\mathbb{R}^{n-1}$ of $-\Delta' v = g(v)$. Here Δ' is the Laplacian in $n-1$ variables and $x_2, \ldots, x_n$ are the coordinates used for $\mathbb{R}^{n-1}$. Moreover, v is even in x_j, v is decreasing in x_j for $x_j \geq 0$, given $\varepsilon > 0$, there exists $K > 0$ such that $v(x) \leq \varepsilon$ if $x_j \geq K$ for $2 \leq j \leq n$ and $v(0) \geq \frac{1}{2}a_1$. If $\lim_{t \to \infty} v(te_j) = 0$ for $2 \leq j \leq n$, the various decreasing properties of v imply that $v(x) \to 0$ as $\|x\| \to \infty$ and we again obtain a contradiction by a result below. If there exists j with $2 \leq j \leq n$ such that $\lim_{t \to \infty} v(te_j) \neq 0$, then, as before and as in [12], one finds that $w(x) = \lim_{x_j \to \infty} v(x)$ is a solution of a similar equation on $\mathbb{R}^{n-2}$ and

w satisfies similar properties. We can repeat the process until we get a solution on a lower dimensional space which tends to zero as $\|x\| \to \infty$. Once again, the result at the end of the proof of (ii) will imply that this case does not occur. Thus we have shown that the latter case does not occur.

Hence the former case holds. Since a ball of radius $\varepsilon^{1/2}\overline{\lambda}^{-1/2}$ has much smaller curvature than ∂D, we can move the ball centre x_0 right up to the boundary. More precisely, if $x_1 \in \partial D$, we can choose a ball $\widetilde{B}$ with centre $\widetilde{x}_1$ on the normal to ∂D at x_1 of radius $\varepsilon^{1/2}\overline{\lambda}^{-1/2}$ with $\widetilde{B} \subseteq D$ and $x \in \partial\widetilde{B}$. Thus, by our sweeping family of subsolutions, a solution u will satisfy $u \geq \phi(\varepsilon^{-1/2}\overline{\lambda}^{1/2}(x - \widetilde{x}_1))$ on $\widetilde{B}$. If $\overline{\lambda}$ is large, we can use the asymptotics in §2 of [9] to estimate ϕ. Using this we find that $u \geq z_0(\varepsilon^{-1/2}t) - \delta$ close to ∂D (where $x = s + tn(s)$). Since our estimates also imply that $u \geq a_1 - \delta$ away from ∂D, we have good lower estimates of solutions u in all of D for small ε.

To obtain estimates for solutions u in the interior of D, we can now repeat the arguments in [12] as refined in the first part of the proof of Proposition 1 here. The main point is that, whenever we use a blow up argument away from $\partial\Omega$, we will obtain a function $\geq a_1$ in $\mathbb{R}^n$. We can apply our previous ideas to $u - a_1$ (a is replaced by $a_2 - a_1$). (For example, in Lemmas 2 and 3 in [12], we consider ℓ or ℓ_1 larger than a_1.) The two cases we have to treat slightly differently are the case where $u_i(t_j e_j) = a_2$ and $\varepsilon_i^{-1/2} d(t_j e_j, \partial D)$ is bounded for all i (where the blow up gives a half space problem) and in part of the proof of Proposition 1 here where we sometimes end up after a blow up with a half space problem. (The half space cases are different because we necessarily have points near the boundary where $\widetilde{u} < a_1$. Here $\widetilde{u}$ is the function after the blowing up.) We consider these two cases separately.

In the first case, if we use a blow up argument as in the proof of Lemma 3 in [12], we obtain a bounded positive solution $\widetilde{u}$ of $-\Delta u = g(u)$ on a half space $T = \{x \in \mathbb{R}^n : x_1 \geq 0\}$ such that $\widetilde{u} = 0$ on ∂T, $\widetilde{u}$ is strictly increasing in x_1, $\widetilde{u}$ is even in x_j for $j \geq 2$, $\widetilde{u}$ is decreasing in x_j for $j \geq 2$ and $x_j \geq 0$ and $\lim_{x_1 \to \infty} \widetilde{u}(x) \geq a_1$ on $\mathbb{R}^{n-1}$. (The last result comes from the lower estimate for u above.) We also find $\widetilde{u} > a_2$ somewhere (since $\widetilde{u}$ is strictly increasing in x_1 and $\widetilde{u} = a_2$ somewhere by the blow up construction). Moreover, by the same proof as in Lemma 4 in [12], given $\mu > a_1$, there exists $K > 0$ such that $\widetilde{u}(x) \leq \mu$ if $|x_j| \geq K$ for $2 \leq j \leq n$. We can then show this is impossible by varying slightly the proof of Lemma 4(ii) in [12] which is an inductive proof (on n). At any stage in the proof there where we reduce to a full space problem, we can apply Lemma 4 in [12] to $\widetilde{u} - a_1$ and obtain a contradiction. We can use the argument there to reduce to the case where, given $\mu > a_1$, there is a $K > 0$ such that $\widetilde{u}(x) \leq \mu$ if $|x_j| \geq K$ for some $j \geq 2$. There is one case where the proof needs to be changed slightly. If we let $\rho(x) = \lim_{x_2 \to \infty} u(x)$, then as in [12], ρ satisfies a similar equation in one lower dimension and has similar properties. However, it is possible that $\rho(x) \leq a_2$ on T while $\rho(x) > a_1$ somewhere. We can show that this case is impossible by the argument in the next paragraph below. Let $w(x) = \lim_{x_1 \to \infty} \widetilde{u}(x_1, x')$. By applying the argument in the proof of Lemma 4 in [12] as improved in the proof of Proposition 1 here to $\widetilde{u} - a_1$, there exist $\alpha < 0$ and an exponentially decreasing positive function $\widehat{\phi}$ on $\mathbb{R}^{n-1}$ such that

$$-\Delta' \widehat{\phi} - g'(w(x'))\widehat{\phi} = \alpha \widehat{\phi}$$

on $\mathbb{R}^{n-1}$ where Δ' is the Laplacian on $\mathbb{R}^{n-1}$. As there, the first and second derivatives of $\widehat{\phi}$ also decay exponentially. We can then complete the proof much as in the corresponding proof in Lemma 4 in [12] (with $m = 1$). To see that $\widetilde{u}$ converges uniformly to w, we use the lower estimates for u near $\partial\Omega$ obtained by the subsolutions to ensure that $\widetilde{u} \geq a_1 - \delta$ if x_1 is large uniformly in x' and the various increasing and decreasing properties of $\widetilde{u}$ and w. (We really should, as in [12], allow the case where $\widetilde{u}$ is increasing in more than one variable but we can easily reduce to the present case.) This completes the proof of this case.

The other time when we can end up with a half space problem is in the argument in the first paragraph of the proof of Proposition 1 here. If we follow the argument there, we end up with a positive solution z of $-\Delta u = g(u)$ on the half space $T = \{x \in \mathbb{R}^n : x_1 \geq 0\}$ such that z is increasing in x_1, even in x_j for $j \geq 2$, decreasing in x_j for $x_j \geq 0$ and $j \geq 2$, $z(\widetilde{\tau}, 0) = \frac{1}{2}(a_1 + a_2)$ for some $\widetilde{\tau} > 0$. Moreover, as there, $z \leq a_2$ on T. (Note that $\frac{1}{2}(a_1 + a_2)$ is the analogue of $\frac{1}{2}a$ and a_2 is the analogue of a.) As there, if we define $\widetilde{z}(x_1) = z(x_1, 0)$, we find that $-\widetilde{z}'' \leq g(\widetilde{z})$ on $x_1 \geq 0$. Moreover, $\widetilde{z}(0) = 0$ and $\widetilde{z}$ is increasing in x_1. If $x_1 \geq \tau$, then $\widetilde{z}(x_1) \geq \widetilde{z}(\widetilde{\tau}) > a_1$ and $\widetilde{z}(x_1) \leq a_2$. Thus $g(\widetilde{z}(x_1)) \leq 0$. Hence we see that $\widetilde{z}$ is increasing and convex for $x_1 \geq \widetilde{\tau}$. Moreover, $\widetilde{z}''(x_1) \geq -g\big(\frac{1}{2}(a_1 + a_2)\big) > 0$. Hence we have a contradiction since $\widetilde{z}$ is bounded. This shows that this case does not occur.

Hence we can repeat the arguments in [12] and deduce that a solution u is uniformly close to some S_{ε, u_0} uniformly on compact subsets of D. (Remember that $u \geq a_1 - \delta$ on such sets.) By what we have already proved and the decreasing properties of a solution u, to complete the proof of (ii), it suffices to establish upper estimates for u near ∂D. These estimates follow from a by now standard blowing up argument since the maximal positive solution of $-\Delta u = g(u)$ in T, $u > 0$ in T, $u = 0$ on ∂T, $u \leq a_1$ in T is $z_0(x_1)$. There are many ways to see this (cp. [10] for related results). One way is to use that if $u(x_1, x')$ is a solution, then $\sup_{x' \in \mathbb{R}^{n-1}} u(x_1, x')$ is a subsolution and a_1 is a supersolution and thus there is a solution v between them which is a function of x_1 only. By analysing the differential equation by using the first integral, one easily sees that $v = z_0$ and our claim follows.

This completes the proof of the first statement of part (ii) except that we have still to prove that $-\Delta u = g(u)$ has no positive solutions $\widetilde{u}$ such that $\widetilde{u}(x) \to 0$ as $\|x\| \to \infty$, $\widetilde{u}$ is even in x_i and $\widetilde{u}$ is decreasing in x_i for $x_i \geq 0$. This is by an easy averaging argument. We let

$$\widehat{u}(r) = \int_S \widetilde{u}(rw)\, dw$$

where S is the unit sphere. Thus $\widehat{u}$ is decreasing on $[0, \infty)$ and $\widehat{u} \to 0$ as $r \to \infty$. By integrating the equation for $\widetilde{u}$ over the sphere of radius r, we have

$$\begin{aligned} -L\widehat{u}(r) = \int_S f(\widetilde{u}(rw))\, dw &\geq c \int_S (\widetilde{u}(rw))^q\, dw \qquad \text{for large } r \\ &\geq \widetilde{c}(\widehat{u}(r))^q \qquad \text{by Jensen's inequality.} \end{aligned}$$

We can obtain a contradiction by integrating this differential inequality for large r by a simple modification of the arguments in Toland [36] or Gidas [21].

This completes the proof of (ii) except to note that the existence of at least one solution of this type for small ε follows by a simple degree argument since the maximal solution has index 1 (since it is non-degenerate and stable) and since the sum of the indices of the positive solutions u with $\|u\|_\infty \in (a_1, a_3)$ must be zero (because it is easy to see that there are no such positive solutions if ε is large).

(iii) By (i) and (ii) and by our earlier results on the uniqueness of u_0 it suffices to prove the uniqueness of the positive solutions near S_{ε,u_0} if ε is small. Suppose by way of contradiction that u_i and v_i are positive solutions of (1) both uniformly close to S_{ε_i,u_0} where $\varepsilon_i \to 0$ as $i \to \infty$. As usual, we see that $h_i = (\|u_i - v_i\|_\infty)^{-1}(u_i - v_i)$ is a solution of

$$\begin{aligned} -\varepsilon_i \Delta h &= g'(\theta(x))h \qquad \text{in } D, \\ h &= 0 \qquad \text{on } \partial D, \end{aligned} \tag{10}$$

where $\theta(x)$ is between u_i and v_i and thus is uniformly close to S_{ε_i,u_0}. Let us consider where h has a positive maximum at x^i. (A similar argument would work for a negative minimum.) By the maximum principle, $g'(\theta(x^i)) \geq 0$. We prove that x_i is very close to either the boundary or the centre. Let us first prove that $\theta(x^i)$ cannot be close to a_1. By the above and our assumptions on g, this is obvious unless $\theta(x^i) = a_1$. Let $T = \{x : \theta(x) = a_1 \text{ and } h(x) = \|h\|_\infty\}$. This lies in the interior of D (since $u_i = v_i = 0$ on ∂D). Choose an open neighborhood W of T in D with smooth boundary such that $\theta(x)$ is close to a_1 on $\overline{W}$ and $h(x) > 0$ on $\overline{W}$. (Note that W need not be connected.) Let W_1 be a component of W. Since $g'(y) \leq 0$ close to a_1, we see from the equation that $-\Delta h \geq 0$ on W_1. Now the maximum of h on W_1 occurs in the interior. Thus by the maximum principle h is constant on W_1 (and hence $g'(\theta(x)) = a_1$ on W_1). Hence $W_1 \subseteq T$. This contradicts our choice of W_1 and so $\theta(x^i) \neq a_1$. Moreover, $\theta(x^i)$ is not close to

a_1 since $g'(\theta(x^i) \geq 0.)$ By Theorem 3(ii), it follows that x^i must be at a distance of at most $K\varepsilon_i^{1/2}$ from $\{0\} \cup \partial D$. We handle the two possibilities separately.

If the distance of x^i from ∂D is of order $K\varepsilon_i^{1/2}$, a by now standard blowing up argument ensures that we have a non-trivial bounded solution of

$$\tag{11} \begin{aligned} -\Delta h &= g'(z_0(x_1))h \qquad \text{in } T, \\ h &= 0 \qquad \text{in } \partial T, \end{aligned}$$

where $T = \{x \in \mathbb{R}^n : x_1 \geq 0\}$. Here we use that u_i and v_i are uniformly close to $z_0(x_1)$ near ∂D (near in the scaled variables). h is bounded because $\|h_i\|_\infty = 1$ and h is non-trivial because in the scaled variables, x^i is at a uniformly bounded distance from the boundary. This is impossible by Proposition 2 in [9] and the remark after it.

If the distance of x_i from zero is of order $K\varepsilon_i^{1/2}$, we can use a similar blowing up argument to obtain a bounded non-trivial solution of

$$\tag{12} -\Delta h = g(u_0(r))h \qquad \text{in } \mathbb{R}^n.$$

Moreover, h is even in each x_i because u_i and v_i are. If $n = 1, 2$, this is impossible by the proof of Theorem 2(iii). If $n \geq 3$, we can use the same argument provided we prove that $h \to 0$ as $\|x\| \to \infty$. We will prove this, which will complete the proof. We choose $c_1 < d_1 < a_1 < d_2 < c_2$ such that $g'(y) \leq 0$ on $[c_1, c_2]$. Choose $t_0 > 0$ such that $z(t_0) = d_1$. We prove that h_i is uniformly small on $T_i = \{s + \varepsilon_i^{1/2} t_0 n(s) : s \in \partial D\}$. If not, in our earlier blowing up argument near the boundary we have (after rescaling) $\|\widetilde{h}_i\|_\infty = 1$ (where $\widetilde{h}_i$ is h_i rescaled) and $\widetilde{h}_i$ is not small at a point at bounded distance from the boundary. When we blow up we obtain a non-trivial solution of (11), which is impossible. Thus h_i is uniformly small on T_i. Choose μ_2 such that $u_0(\mu_2) = d_2$ and let $S_i = \{x \in \mathbb{R}^n : \|x\| = \varepsilon_i^{1/2}\mu_2\}$. Note that on S_i, $\theta(x) < c_2$ (since $\theta(x)$ is close to S_{ε_i, u_0}). We consider (10) on the set W_i between S_i and T_i. Note that $g'(\theta(x)) \leq 0$ on W_i and that $v_i(x) = \delta_i + c_i\|x\|^{2-n}$ is harmonic on W_i. We choose δ_i small so that $h_i(x) \leq \delta_i$ on T_i and note that if c_i is chosen appropriately of order $\varepsilon_i^{(n-2)/2}$, then $v_i(x) \geq h_i$ on S_i (since $\|h_i\| = 1$). Thus $v_i \geq h_i$ on ∂W_i. By the maximum principle, it follows that $v_i \geq h_i$ on W_i. (If $h_i - v_i$ has a positive maximum in W_i, then $h_i > 0$ there and $-\Delta(h_i - v_i) = \varepsilon_i^{-1} g'(\theta_i)h_i \leq 0$ nearby. We then obtain a contradiction by a similar argument to one at the beginning of this part.) Similarly, $-h_i \leq v_i$. Hence we find that $|h_i(x)| \leq \delta_i + \widetilde{c}_i\|\varepsilon_i^{-1/2}x\|^{2-n}$ on W_i where $\widetilde{c}_i$ is of order 1 and δ_i is small. Hence with the usual rescaling, $|\widetilde{h}_i(x)|$ is small away from the origin if $n \geq 3$ (where $\widetilde{h}_i$ is h_i in the scaled variables). By the usual limit argument we find a solution $\widetilde{h}$ of (12) on $\mathbb{R}^n$ with $|\widetilde{h}(x)| \to 0$ as $\|x\| \to \infty$. As we noted earlier, this suffices to complete the proof.

REMARKS.

1. The remarks after Theorem 2 have analogues here.
2. It seems likely that the weakened assumptions mentioned after the statement of Theorem 3 can be further improved by replacing $3 \le m \le n$ by $m = n$ and replacing $u(0) \in (0, a_3)$ by $u(0) \in (a_2, a_3)$. On the other hand, one can easily construct examples where (2) has positive radial solutions u_1 such that $u_1(0) \in (a_2, a_3)$ and $u_1(r) \to 0$ as $r \to \infty$ and where g satisfies all the assumptions of Theorem 3(i) except that $g(y) \sim y^q$ as $y \to 0$. Here $q > n(n-2)^{-1}$ (and any such q can occur). It follows (with care) in this case that (1) has solutions of rather different type to those in Theorem 3(ii) for all small ε when D is a ball. Thus Theorem 3(ii) is no longer true in this case.
3. We suspect that our methods can be adapted to handle at least partially cases where g changes sign more times. It seems likely that it is still true that positive solutions have at most 2 layers (for ε small and domains of type R_n).

3. Counterexamples for more general domains

In this section,we produce a number of counterexamples showing that our main results are false for suitable dumbbells. This shows, as we conjectured in [12], that good results are only true for domains with considerable restrictions on their geometry. (It would be interesting to understand the case of a general convex domain.) We construct counterexamples for the simplest problem of this type, that is,

$$(13)\qquad \begin{aligned} -\varepsilon\Delta u &= u^p - u && \text{in } D,\\ u &= 0 && \text{on } \partial D, \end{aligned}$$

where $1 < p < (n-2)^{-1}(n+2)$ and $n \ge 2$. This was the main problem in [12]. One can then deduce counterexamples in other cases from this. We consider D_δ, a dumbbell with cylindrical joining strip of radius δ and length $2k_0$. We will show that if δ is chosen suitably (and the ends and the joining strip are chosen suitably), then (13) has at least 2 positive solutions (in fact, at least 3 positive solutions) for all small positive ε. Note that the issue here is the existence for all small ε. For fixed ε, it is easy to prove the result by domain variation arguments (as in [14]). The result we want here is much more delicate. (In fact, the gap in the energy between the solutions we want and others is exponentially small in ε.)

More precisely, we consider a dumbbell shaped domain D_δ with a long thin neck. Here the two balls B_1, B_2 have radius 1 and we have smoothed the corners

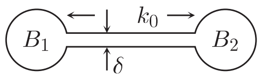

Figure 1

to retain the smoothness. We will prove that for all small positive ε the solution which minimizes $E = \int_{D_\delta} (\varepsilon|\nabla u|^2 + u^2)$ subject to the constraint

$$\int_{D_\delta} |u|^{p+1} = 1 \tag{14}$$

(for $u \in W^{1,2}(D_\delta)$) is not symmetric in x_1 provided that δ is small and k_0 is fixed. Here x_1 is the coordinate along the strip. This proves our claim because it is easy to see that this solution $\widetilde{u}$ (when rescaled) gives a positive solution of (13) and $\widetilde{u}(-x_1, \widehat{x}_1)$ is also a positive solution. (Here $(x_1, \widehat{x}_1)$ are the coordinates for $\mathbb{R}^n$ with $\widehat{x}_1 \in \mathbb{R}^{n-1}$.) Let M_δ be the minimum value of this variational principle. Let M_δ^s be the corresponding minimum value in the space of functions even in x_1. We will prove that $M_\delta^s > M_\delta$ (for δ small and $0 \le \varepsilon < \widehat{\mu}$ for some $\widehat{\mu}$). This suffices to prove our claim. Unfortunately, the proof of this tends to be quite technical and delicate (because $M_\delta^s - M_\delta$ is exponentially small in ε).

We start by estimating M_δ. (This is the easier part of the argument.) Let u_0 be the unique positive radial solution of $-\Delta u + u = u^p$ on $\mathbb{R}^n$ such that $u(x) \to 0$ as $\|x\| \to \infty$. Up to translation, this is the unique positive solution of this equation such that $u(x) \to 0$ as $r = \|x\| \to \infty$. It is well known that u_0 decays exponentially and in fact, by using a change of variable $u_0(r) = r^{-(n-1)/2} v(r)$ and by using standard results for linear ordinary differential equations, we readily find that $\lim_{r\to\infty} r^{(n-1)/2} u_0(r) \exp(r) = \mu$ exists, $\mu \in (0,\infty)$ and the corresponding estimate for u_0' holds. Note that a lower estimate for M_δ is immediate since $M_\delta \ge \overline{M}$ where $\overline{M}$ is the minimum value of the corresponding problem on $\mathbb{R}^n$. By a simple rescaling, we see that $\overline{M} = \varepsilon^\beta Q$ where Q is the infimum of $\int_{\mathbb{R}^n} (|\nabla u|^2 + u^2)$ over $u \in W^{1,2}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} u^{p+1} = 1$ and $\beta = -\frac{1}{2} n(p-1)(p+1)^{-1}$. Since this infimum must be achieved by a scalar multiple of u_0 (by the uniqueness), we easily see that $Q = \big(\int_{\mathbb{R}^n} u_0^{p+1}\big)^{(p-1)/(p+1)}$. Note that one calculates $\int_{\mathbb{R}^n} (|\nabla u_0|^2 + u_0^2)$ by multiplying the equation satisfied by u_0 by u_0. Choose a smooth function ϕ of compact support in the right hand ball and radial about the centre of this ball so that $\phi = 1$ if $\|x\| \le K$. To obtain an upper estimate for M_δ, we use the test function $C\phi(x) u_0(\varepsilon^{-1/2}(x - x_0))$ where x_0 is the centre of the right hand ball and C is chosen so that the constraint (14) is satisfied. By a rather tedious calculation with this test function (and using the estimates for u_0), one finds that

$$M_\delta \le \varepsilon^\beta (Q + C_1 e^{-2\mu\varepsilon^{-1/2}}) \tag{15}$$

where $C_1 > 0$ and is independent of ε. A few points on the calculation. Firstly, it is easier to do the calculations by rescaling the problem to one on domains $\varepsilon^{-1/2}D_\delta$, and secondly, the higher order corrections in the calculation of C will not affect the first order term correction term in (15) since $1+p>2$.

We now obtain an estimate for M_δ^s. Firstly, note that by choosing a symmetric function with peaks in both balls (suitably normalized), one easily shows that $M_\delta^s \le 2^{p(p+1)^{-1}}Q\varepsilon^\beta$ (uniformly in δ and k_0 for small ε). As remarked in [12] (cp. Remark 3 after Theorem 1 here) we can find a solution ϕ_ε of (13) which is even in x_j for all j and has its maximum near the centre by an implicit function argument. We prove that, if $M_\delta^s < 2^{(p-1)(p+2)^{-1}}M_\delta$, then ϕ_ε (rescaled) minimizes M_δ^s (for small ε). Once again it is more convenient to work on the rescaled domain $\widetilde{D}_\varepsilon = \varepsilon^{-1/2}D_\delta$. We prove first that the minimizer ψ_ε of M_δ^s is uniformly bounded in ε (in the L^∞ norm). This follows by standard blowing up arguments (cp. Lemma 1 in [12] and the remarks after it) if we note that, by our remarks above,

$$(16)\quad \widehat{M}_\varepsilon^s = \widehat{M}^s(\widetilde{D}_\varepsilon) = \inf\Big\{\int_{\widetilde{D}_\varepsilon}(|\nabla u|^2+u^2) : u \in W_0^{1,2}(\widetilde{D}_\varepsilon), \int_{\widetilde{D}_\varepsilon} u^{p+1} = 1,\ u \text{ is symmetric in } x_1\Big\}$$

is bounded below by Q and above by $2^{p(p+1)^{-1}}Q$ where $Q>0$ and if we note that the minimizer ψ_ε satisfies

$$-\Delta\psi_\varepsilon + \psi_\varepsilon = \widehat{M}_\varepsilon^s \psi_\varepsilon^p$$

in $\widetilde{D}_\varepsilon$. Hence $(\widehat{M}_\varepsilon^s)^{1/(p-1)}\psi_\varepsilon$ solves $-\Delta u + u = u^p$ in $\widetilde{D}_\varepsilon$. For future reference we note that $\widehat{M}_\varepsilon$ and $\widehat{M}(\widetilde{D}_\varepsilon)$ are defined analogously except that we drop the symmetry in x_1.

Secondly, note that the Gidas–Ni–Nirenberg theory [19] applied to D_ε implies that any point $\widehat{x}_\varepsilon$ where ψ_ε attains its maximum, must satisfy $d(\widehat{x}_\varepsilon, \partial\widetilde{D}_\varepsilon) \to \infty$ as $\varepsilon \to 0$. Now assume $\varepsilon_i \to 0$ and $x^i \in \widetilde{D}_{\varepsilon_i}$ such that $\psi_{\varepsilon_i}(x^i) \ge \widetilde{\tau} > 0$ for all i and $d(x^i, \partial\widetilde{D}_{\varepsilon_i}) \to \infty$ as $i \to \infty$. (For example we could choose $x^i = \widehat{x}_{\varepsilon_i}$.) If we shift the origin to x^i, a by now standard argument ensures that a subsequence ψ_{ε_i} (rescaled) converges uniformly on compact sets to a bounded positive solution $\overline{\psi}$ of $-\Delta u = Su^p - u$ in $\mathbb{R}^n$ where $S \ge Q$. Moreover, $\overline{\psi} \in W^{1,2}(\mathbb{R}^n)$. To see the latter result, note that, if $R>0$ and i is large (depending on R), then

$$\int_{B_R}(|\nabla\psi_{\varepsilon_i}(x-x^i)|^2 + (\psi_{\varepsilon_i}(x-x^i))^2) \le \widehat{M}_{\varepsilon_i}^s \le 2^{p(p+1)^{-1}}Q.$$

Hence

$$\int_{B_R}(|\nabla\overline{\psi}|^2 + \overline{\psi}^2) \le 2^{p(p+1)^{-1}}Q$$

and our claim follows. Since $\overline{\psi} \in L^2$ and since $\overline{\psi}$ is uniformly continuous (by the regularity theory for elliptic equations), it follows that $\overline{\psi}(x) \to 0$ as $\|x\| \to \infty$. Thus, by [29], $\overline{\psi}$ is a translate of the unique radial solution $S^{-1/(p-1)}u_0$. Hence we see that if there are two points $(\pm x_1, \widehat{x}_1)$ in $\widetilde{D}_{\varepsilon_i}$ both far from $\partial\widetilde{D}_{\varepsilon_i}$ and both far apart, where ψ_{ε_i} is not small, then

$$\begin{aligned}\widehat{M}^s_{\varepsilon_i} &\geq \int_{\widetilde{D}_{\varepsilon_i}} (|\nabla\psi_{\varepsilon_i}|^2 + \psi^2_{\varepsilon_i}) \geq S^{-2/(p-1)}(2-\widehat{\delta}) \int_{\mathbb{R}^n} (|\nabla u_0|^2 + u_0^2) \\ &\geq (2-\widehat{\delta}) S^{-2/(p-1)} Q^{(p+1)(p-1)}\end{aligned}$$

where $\widehat{\delta}$ is small. By a simple rescaling and by (15) we see that this contradicts our assumption that $M^s_\delta < 2^{(p-1)/(p+2)} M_\delta$ (for $\widehat{\delta}$ small). (Note that S is the limit through a subsequence of $\widehat{M}^s_{\varepsilon_i}$ and hence $\widehat{M}^s_{\varepsilon_i}$ is close to S and that our inequality is equivalent to the corresponding one involving $\widehat{M}^s_{\varepsilon_i}$ and $\widehat{M}_{\varepsilon_i}$.) Since $x_j \partial u/\partial x_j < 0$ if $j \geq 2$ and $x_j > 0$ by Gidas–Ni–Nirenberg and since ψ_{ε_i} are even in x_1, it follows easily that ψ_{ε_i} is uniformly small except near the "centre" of $\widetilde{D}_{\varepsilon_i}$. By near, we mean within a bounded distance. From this and since ψ_{ε_i} is even in each x_j, we see that ψ_{ε_i} must be uniformly close to u_0. This and the local uniqueness in Remark 3 after Theorem 1 in [12] implies our claim on ψ_{ε_i}. (Note that since every convergent subsequence has the same limit, the whole sequence must converge.)

We now obtain good estimates for ψ_{ε_i} away from the centre. Assume $0 < \tau < 1$ and $0 < k_1 < k_0$. Note that we have already shown that ψ_{ε_i} (rescaled) is uniformly close to u_0 and hence $|\psi_{\varepsilon_i}| \leq (1-\tau_1^2)^{1/p}$ if $x \in \widetilde{D}_{\varepsilon_i}$ and $\|x\| \geq K$ (for suitable K) (where $\tau < \tau_1 < 1$). Hence, if $K \leq x_1 \leq \varepsilon_i^{-1/2}k_1$, $\Delta u = a(x)u$ where $\tau^2 < \tau_1^2 \leq a(x) \leq 1$ and where $u = \psi_{\varepsilon_i}$. We now use estimates for linear equations to estimate u on this part of the strip. In particular, we show that ψ_{ε_i} is very small near $x_1 = \frac{1}{2}\varepsilon_i^{-1/2}k_1$. As a comparison function, we use $z = \phi_1(\varepsilon_i^{1/2}\delta^{-1}\widehat{r})\cosh\bigl(\tau\bigl(x_1 - \frac{1}{2}\varepsilon_i^{-1/2}k_1\bigr)\bigr)$ (which solves $\Delta v \geq \tau_1^2 v$) for suitable τ close to 1 and ϕ_1 the first eigenfunction for the ball of radius 2 in $\mathbb{R}^{n-1}$. Here $\widehat{r} = \|\widehat{x}_1\|$. Now the maximum of $z^{-1}\psi_{\varepsilon_i}$ on $\{(x_1,\widehat{x}_1) : K \leq x_1 \leq \varepsilon^{-1/2}k_1,\ \|\widehat{x}_1\| \leq \varepsilon_i^{-1/2}\delta\}$ occurs on the boundary by Theorem 11 in [34] and clearly does not occur when $\|\widehat{x}_1\| = \varepsilon_i^{-1/2}\delta$ (where $\psi_{\varepsilon_i} = 0$). Since ψ_{ε_i} is uniformly bounded, it follows that $|\psi_{\varepsilon_i}| \leq K_1\bigl(\cosh\bigl(\frac{1}{2}\tau\varepsilon_i^{-1/2}k_1\bigr)\bigr)^{-1}$ if x_1 is close to $\frac{1}{2}\varepsilon_i^{-1/2}k_1$ (to within order 1) and $x \in \widetilde{D}_{\varepsilon_i}$. (A similar argument appears on p. 665 of [8].) Now we can use elliptic regularity theory to obtain the corresponding estimate for $\nabla\psi_{\varepsilon_i}$ (at the expense of shrinking the length of the order 1 segment). Since ψ_{ε_i} is small away from the centre, we see that $-\Delta\psi_{\varepsilon_i} = b\psi_{\varepsilon_i}$ on $T_i = \bigl\{x \in \widetilde{D}_{\varepsilon_i} : x_1 \geq \frac{1}{2}\varepsilon_i^{-1/2}k_1\bigr\}$ where $b < 0$. Thus by the maximum principle, ψ_{ε_i} on T_i must have its maximum on the boundary and hence we see that $|\psi_{\varepsilon_i}(x)| \leq \widetilde{K}_1 \exp\bigl(-\frac{1}{2}\tau\varepsilon_i^{-1/2}k_1\bigr)$ on T_i. As before we can deduce the corresponding estimate for the derivative. Now choose $\phi_i : \mathbb{R} \to [0,1]$ smooth such that $|\phi_i'| \le K$ on $\mathbb{R}$ for all i, $\phi_i(t) = 1$ if $|t| \le \frac{1}{2}\tau\varepsilon_i^{-1/2}k_1$, and $\phi_i(t) = 0$ if $|t| \ge \frac{1}{2}\tau\varepsilon_i^{-1/2}k_1 + 1$. Then $Y_i = c_i\phi_i(x_1)\psi_{\varepsilon_i} \in W_0^{1,2}(S_i)$ where $S_i = \{(x_1, \widehat{x}_1) : |x_1| \le \tau\varepsilon_i^{-1/2}k_1,\ \|\widehat{x}_1\| \le \delta\varepsilon_i^{-1/2}\}$ and c_i is chosen (very close to 1) such that $c_i\phi_i(x_1)\psi_{\varepsilon_i}$ satisfies our constraint. By our above decay estimates for ψ_{ε_i} and some elementary but tedious calculations, we find that

$$\int_{S_i} ((\nabla Y_i)^2 + Y_i^2) \le M_\delta^s + K_2 \exp(-\tau\varepsilon_i^{-1/2}k_1)\varepsilon_i^{-n/2}.$$

(Here we use that ψ_{ε_i} minimizes the variational principle (16) on $\widetilde{D}_{\varepsilon_i}$ for $\varepsilon = \varepsilon_i$.) Thus

$$\widehat{M}(S_i) \le \widehat{M}^s(S_i) \le M_\delta^s + K_2\varepsilon_i^{-n/2}\exp(-\tau\varepsilon_i^{-1/2}k_1).$$

By symmetrization (cp. Bandle [1]), $\widehat{M}(B_{\alpha(i)}) \le \widehat{M}(S_i)$ where $B_{\alpha(i)}$ has the same volume as S_i, that is, $c_n(\alpha(i))^n = \widetilde{c}_n\varepsilon_i^{-n/2}\delta^{n-1}$. Thus $\alpha(i) \sim \varepsilon_i^{-1/2}\delta^{1-1/n}$. Note that $\alpha(i)$ also depends on k_0 but we have suppressed this dependence. Hence, if $M_\delta^s = M_\delta$, and if we use (15) (or more strictly its analogue on $\widetilde{D}_\varepsilon$), we find that

$$\widehat{M}(B_{\alpha(i)}) \le Q + K_3\exp(-2\mu\varepsilon_i^{-1/2}K) + K_4\varepsilon_i^{-n/2}\exp(-\tau\varepsilon_i^{-1/2}k_1).$$

We show that this last inequality is impossible if δ is small by showing that $\widehat{M}(B_R) \ge Q + C_2\exp(-aR)$ for large R if $a > 2$ (and thus $\widehat{M}(B_{\alpha(i)}) - Q \ge C_2\exp(-aC_3\varepsilon_i^{-1/2}\delta^{1-1/n})$). This will complete the proof. Note that by symmetrization, the minimizer of $\widehat{M}(B_R)$ must be a radial function and hence a solution of an ordinary differential equation. Let u_R denote the unique (by [27]) positive radial solution of

$$-\Delta u = u^p - u, \qquad u(R) = 0, \qquad u'(0) = 0.$$

Then the minimizer of $\widehat{M}(B_R)$ is αu_R where α is chosen to satisfy the constraint. By an easy computation similar to earlier, one finds that

$$\widehat{M}(B_R) = \left(\int_{B_R} u_R^{p+1}\right)^{1-2/(p+1)}.$$

Since $\widehat{M}$ approaches $Q > 0$ as $R \to \infty$, it follows that $(d/dR)\widehat{M}(B_R)$ is of the form

$$\gamma_R\frac{d}{dR}\left(\int_0^R r^{n-1}u_R^{p+1}(r)\,dr\right) \qquad \text{where } \gamma_R \to \widetilde{\gamma} > 0 \text{ as } R \to \infty.$$

Since $u_R(R) = 0$, it follows that for large R,

$$\frac{d}{dR}\widehat{M}(B_R) \sim (p+1)\widehat{\gamma}\int_0^R r^{n-1}(u_R(r))^p\frac{du_R(r)}{dR}\,dr \tag{17}$$

for large R. Note that to check that u_R is a differentiable function of R, one rescales back to the unit interval and uses the implicit function theorem. Note that it is proved in [27] (combined with arguments in [12]) that u_R is a non-degenerate solution of $-\Delta u = u^p - u$ in B_R, $u = 0$ on ∂B_R in the space of radial functions. By differentiating the equation for u_R in R, we see that du_R/dR is a solution of

$$-Lw = (pu_R^{p-1} - 1)w, \qquad w'(0) = 0. \tag{18}$$

(To see the last condition, recall that $u_R'(0) = 0$ for all R where $'$ denotes derivatives in r.) Thus, by the uniqueness of the initial value problem,

$$\frac{\partial u_R}{\partial R} = \mu_R v_R \tag{19}$$

where v_R is the solution of (18) satisfying $v_R(0) = 1$. Moreover, by differentiating $u_R(R) = 0$ in R, we see that

$$\frac{\partial u_R}{\partial R}(R) + \frac{\partial u_R}{\partial r}(R) = 0. \tag{20}$$

This ensures that

$$\frac{\partial u_R}{\partial R}(R) = -\frac{\partial u_R}{\partial r}(R) \neq 0$$

and hence $\mu_R \neq 0$. By multiplying the equation for u_R by v_R and that for v_R by u_R and subtracting, we find that

$$\begin{aligned}(p-1)\int_0^R u_R^p v_R r^{n-1}\,dr &= \int_0^R v_R \frac{d}{dr}(r^{n-1}u_R') - u_R\frac{d}{dr}(r^{n-1}v_R')dr \\ &= [r^{n-1}(u_R'v_R - u_R v_R')]_0^R = R^{n-1}u_R'(R)v_R(R)\end{aligned} \tag{21}$$

(since $u_R(R) = 0$). By (17), (19), (20) and (21), we see that

$$\frac{d\widehat{M}_R}{dR}(B_R) \sim -R^{n-1}(u_R'(R))^2$$

for large R. Thus we will have completed our proof if we prove that, if $\widetilde{a} > 1$,

$$-R^{(n-1)/2}u_R'(R) \geq e^{-\widetilde{a}R} \tag{22}$$

for large R.

To prove this, we consider u_R on $[\alpha R, R]$ where $0 < \alpha < 1$. We will prove that

$$u_R(\alpha R) \sim R^{-(n-1)/2}\exp(-\alpha R) \tag{23}$$

for large R. Assuming this, we complete the proof. Note that (23) implies that u_R is uniformly small on $[\alpha R, R]$ since u_R is decreasing. Since αR is large, a simple computation shows that $w_R = r^{-(n-1)/2}u_R(r)$ satisfies $w'' + g(r)w = 0$ on $[\alpha R, R]$ where $g(r) \leq 1 + \overline{\delta}$. It follows from Gronwall's inequality (applied to $(w')^2 + w^2$ on $[\alpha R, R]$) that

$$w_R(\alpha R) \leq -w_R'(R)\exp(1+\overline{\delta})(R - \alpha R).$$

Hence by choosing α small, by noting the change of variable from w_R to u_R, and by using the estimate above for $u_R(\alpha R)$, we see that (22) follows.

It remains to prove the estimate (23). By our asymptotics for u_0, it suffices to prove that

$$\sup_{0\le r\le R} |u_R(r) - u_0(r)| \le Ce^{-R}$$

for large R. To see this, we examine a little more closely Remark 3 after the proof of Theorem 1 in [12]. We choose $\phi : \mathbb{R} \to [0,1]$ smooth such $\phi(x) = 1$ if $x \le 0$ and $\phi(x) = 0$ if $x \ge 1$. We then use $\widetilde{u}_R(r) = u_0(r)\phi(r - R + 1)$ as an approximate solution of the equation for u_R. Using our decay properties of u_0 and our choice of ϕ, it is easy to prove that

$$\| -L\widetilde{u}_R + \widetilde{u}_R - (\widetilde{u}_R)^p\|_{\infty,R} \le Ce^{-R}$$

where $\| \ \|_{\infty,R}$ denotes the sup norm on $[0,R]$. (Here we use that $\widetilde{u}_R$ solves the equation exactly except on $[R-1,R]$). Moreover, as in [12] or [8] one can show that, for large R, $-L + (1 - p(\widetilde{u}_R)^{p-1})I$ with boundary conditions $z'(0) = 0$, $z(R) = 0$ is invertible and the inverse is bounded uniformly in R. Hence as in [12] or [8] we can use the contraction mapping theorem to construct a solution $\widehat{u}_R$ of $-Lu = u^p - u$, $u(R) = 0$ such that $\|\widehat{u}_R - \widetilde{u}_R\|_{\infty,R} \le Ce^{-R}$. By the uniqueness of the positive solution, $\widehat{u}_R = u_R$ and our claim follows provided we prove that $\widehat{u}_R > 0$ on $(0,R)$. By the asymptotics for u_0 and by our estimate for $\widehat{u}_R - \widetilde{u}_R$, $\widehat{u}_R > 0$ on $[0,\alpha R]$. On $[\alpha R, R]$, $\widehat{u}_R$ is small and hence $-L\widehat{u}_R(r) = \widetilde{\alpha}(r)\widehat{u}_R(r)$ where $\widetilde{\alpha}(r) < 0$ on $[\alpha R, R]$ (by the equation for $\widehat{u}_R$). Hence $\widehat{u}_R$ has at most 1 zero in $[\alpha R, R]$. Since $\widehat{u}_R(\alpha R) > 0$ and $\widehat{u}_R(R) = 0$, the positivity follows. This proves our claim.

This completes the construction of the example.

Remarks.

1. On domains with non-trivial topology one can more easily obtain a similar result (cp. [1]).
2. These ideas have other uses. For any domain D, some of our ideas can easily be used to construct a mountain pass positive solution of low "energy" for every small positive ε. Moreover, some of our ideas imply that this solution is a "peaked" solution with only one "peak". It would be interesting to understand for which domains all positive solutions for small ε have only a finite number of peaks.
3. These ideas can be used to construct many positive solutions for small ε in the case of annuli. Given a finite subgroup H of $SO(n)$, one can easily construct test functions invariant under H (with a finite number of peaks) where the "energy" is of order ε^{β}. Thus in this symmetric subspace, the minimizer of our constrained problem will have "energy"

$\leq K\varepsilon^{\beta}$ (bounded energy in our usual scaled variables). Hence, if we prove that the minimizer of the constrained problem for (13) in the space of radial functions has larger energy, then we will have proved that for small ε there is a positive H-invariant solution which is not radially symmetric. Thus there are non-radial solutions. In this way, we can frequently construct many non-radial solutions (infinitely many distinct ones if $n = 2$). (This probably can be refined.) This contrasts with the case of large ε where the main result in [15] implies there is a unique positive solution for large ε if the hole in the annulus is small. It remains to prove our claim on the radial solution above. It is easy to see that it suffices to prove that, for small ε,

$$\left(\int_{\varepsilon^{-1/2}a}^{\varepsilon^{-1/2}b} r^{n-1}(u_\varepsilon(r))^{p+1}\,dr\right)^{(p-1)/(p+1)} \to \infty \tag{24}$$

as $\varepsilon \to 0$ where $D = \{x : a < \|x\| < b\}$ and u_ε is the unique positive radial solution of $-\Delta u = u^p - u$ on $\varepsilon^{-1/2}D$ (using [27]). Note that the left hand side of (21) is essentially the "energy" of u_ε for the constrained problem in the scaled variables. If u_ε achieves its maximum at r_ε, Gidas, Ni and Nirenberg [19] implies that $|r_\varepsilon - \varepsilon^{-1/2}a|$ and $|r_\varepsilon - \varepsilon^{-1/2}b|$ tend to infinity at $\varepsilon \to 0$ and a simple blowing up and limiting argument as in [16] implies that $u_\varepsilon(r - r_\varepsilon) - \widetilde{u}_0(r) \to 0$ uniformly on compact sets where $\widetilde{u}_0$ is the positive solution of $-u''(r) = u^p - u$ on $(-\infty, \infty)$, $u \to 0$ as $\to \pm\infty$. Because of the factor r^{n-1} in (24), (24) now readily follows, as required.

Finally, note that these ideas could be combined with those in [16] to obtain information on the Morse index of the radial solutions. If the annulus hole is small, these ideas and those in [16] imply there is global bifurcation of non-radial positive solutions off the branch of positive radial solutions.

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