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RANDOM EQUILIBRIA OF RANDOM GENERALIZED GAMES WITH APPLICATIONS TO NON-COMPACT RANDOM QUASI-VARIATIONAL INEQUALITIES

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Dedicated to Ky Fan

1. Introduction

In the last two decades, the classical Arrow and Debreu result [3] on the existence of Walrasian equilibria has been generalized in many directions. Mas-Colell [35] has first shown that the existence of an equilibrium can be established without assuming preferences to be total or transitive. Next, by using an existence theorem of maximal elements, Gale and Mas-Colell [20] gave a proof of the existence of a competitive equilibrium without ordered preferences. By using Kakutani's fixed point theorem, Shafer and Sonnenschein [43] proved a powerful result on the "Arrow and Debreu Lemma" for abstract economies in which preferences may not be total or transitive but have open graphs. Meanwhile, Borglin and Keiding [8] proved a new existence theorem for a compact abstract economy with KF-majorized preference correspondences. Following their ideas, there have been a number of generalizations of the existence of equilibria for compact abstract economies (see e.g. Aliprantis *et al.* [1], Border [7], Chang [11], Debreu [13], Ding *et al.* [15]–[16], Flam [18], Florenzano [19], Hildenbrand and Sonnenschein [25], Kajii [27], Keiding [28], Mehta and Tarafdar [37], Shafer [42], Khan and Yannelis [29], Mas-Colell and Zame [36], Tian [53], Tan and Yuan [48]–[49], Tarafdar [51], Tarafdar and Mehta [52], Tulcea [54]–[55] etc.). These

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theorems generalized most known equilibrium existence theorems on compact generalized games due to Borglin and Keiding [8], Shafer and Sonnenschein [43], Toussaint [54] and Yannelis and Prabhakar [58].

On the other hand, Debreu discussed the uncertainty of behavior of an economic activity in Chapter 7 of his book [12]. Since then, a series of papers concerning the uncertainty of behavior of economic actions have been published. For example, Hildenbrand [23] considered an economy in which the preferences are random correspondences; Bewley [6] studied the existence of equilibrium in abstract economies with a measure space of agents and with an infinite-dimensional strategy space; Kim *et al.* [30] also proved the existence of equilibria in abstract economies with a measure space of agents and with an infinite-dimensional strategy space by random fixed point theorems.

In this paper, the existence theorems of non-compact random equilibria in which the preference correspondences are L-majorized and constraint correspondences are upper semicontinuous are first obtained. As applications, we give the existence theorems for non-compact random quasi-variational inequalities, which in turn imply several existence theorems for non-compact generalized random quasi-variational inequalities. These results not only generalize the results of Tan [50] and Zhang [59], but also are the stochastic versions of corresponding results in the literature (see e.g. [1]–[13], [16]–[22], [25]–[33], [35]–[46], [47]–[49], [51]–[56] and [58]).

2. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$ and the set of natural numbers is denoted by $\mathbb{N}$. If X is a set, we shall denote by 2^X the family of all subsets of X. Let A be a subset of a topological space X. We shall denote by $\operatorname{int}_X(A)$ the interior of A in X and by $\operatorname{cl}_X(A)$ the closure of A in X. If A is a subset of a vector space, we shall denote by $\operatorname{co} A$ the convex hull of A. If A is a non-empty subset of a topological vector space E and $S, T : A \to 2^E$ are correspondences, then $\operatorname{co} T, T \cap S : A \to 2^E$ are the correspondences defined by $(\operatorname{co} T)(x) = \operatorname{co} T(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively. If X and Y are topological spaces and (Ω, Σ) is a measurable space, and $T : \Omega \times X \to 2^Y$ is a correspondence, the *graph* of T, denoted by $\operatorname{Graph} T$, is the set $\{(\omega, x, y) \in \Omega \times X \times Y : y \in T(\omega, x)\}$ and the correspondence $\overline{T} : \Omega \times X \to 2^Y$ is defined by $\overline{T}(\omega, x) = \{y \in Y : (x, y) \in \operatorname{cl}_{X \times Y} \operatorname{Graph} T(\omega, \cdot)\}$, where for each fixed $\omega \in \Omega, \operatorname{Graph} T(\omega, \cdot) = \{(x, y) \in X \times Y : y \in T(\omega, x)\}$ and $\operatorname{cl} T : \Omega \times X \to 2^Y$ is defined by $\operatorname{cl} T(\omega, x) = \operatorname{cl}_Y(T(\omega, x))$ for each $(\omega, x) \in \Omega \times X$. It is easy to see that $\operatorname{cl} T(\omega, x) \subset \overline{T}(\omega, x)$ for each $(\omega, x) \in \Omega \times X$.

If X and Y are two sets, $A \subset X \times Y$, and $F : X \to 2^Y$, then (1) the domain of F, denoted by $\operatorname{Dom} F$, is the set $\{x \in X : F(x) \neq \emptyset\}$; (2) the projection of A into X, denoted by $\mathrm{Proj}_X A$, is the set $\{x \in X :$ there exists some $y \in Y$ such that $(x, y) \in A\}$; moreover, if both X and Y are topological spaces, (3) F is said to be *lower* (respectively, *upper*) *semicontinuous* if for each closed (respectively, open) subset C of Y, the set $\{x \in X : F(x) \subset C\}$ is closed (respectively, open) in X, (4) $x \in X$ is a *maximal element* of F if $F(x) = \emptyset$ and (5) F is said to be *compact* if for each $x \in X$, there exists a neighborhood V_x of x in X such that $F(V_x) = \bigcup_{x' \in V_x} F(x')$ is relatively compact in Y. Note that $\mathrm{Dom}\, F = \mathrm{Proj}_X \mathrm{Graph}\, F$.

Let X be a subset of a topological vector space E. X is said to have *property* (K) (see [56]) if for each compact subset B of X, the convex hull $\mathrm{co}\, B$ of B is also relatively compact in X.

Let X be a topological space, Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a (single-valued) map and $\phi : X \to 2^Y$ be a correspondence. Then (1) ϕ is said to be of *class* L_θ if for every $x \in X$, $\mathrm{co}\, \phi(x) \subset Y$ and $\theta(x) \notin \mathrm{co}\, \phi(x)$ and for each $y \in Y$, $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X; (2) a correspondence $\phi_x : X \to 2^Y$ is said to be an L_θ-*majorant* of ϕ at $x \in X$ if there exists an open neighborhood N_x of x in X such that (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \mathrm{co}\, \phi_x(z)$, (b) for each $z \in X$, $\mathrm{co}\, \phi_x(z) \subset Y$ and (c) for each $y \in Y$, $\phi_x^{-1}(y)$ is open in X; (3) ϕ is L_θ-*majorized* if for each $x \in X$ with $\phi(x) \neq 0$, there exists an L_θ-majorant of ϕ at x in X. We shall only deal with either the case (I) $X = Y$ and is a non-empty convex subset of a topological vector space and $\theta = I_X$, the identity map on X (in this case, the above notions coincide with the corresponding notions introduced in [58]), or the case (II) $X = \prod_{i \in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j and $X_j = Y$ is a non-empty convex subset of a topological vector space. In both cases (I) and (II), we shall write L in place of L_θ.

A *measurable space* (Ω, Σ) is a pair where Ω is a set and Σ is a σ-algebra of subsets of Ω. If X is a set, $A \subset X$, and $\mathcal{D}$ is a non-empty family of subsets of X, we shall denote by $\mathcal{D} \cap A$ the family $\{D \cap A : D \in \mathcal{D}\}$ and by $\sigma_X(\mathcal{D})$ the smallest σ-algebra on X generated by $\mathcal{D}$. If X is a topological space with topology τ_X, we shall use $\mathcal{B}(X)$ to denote $\sigma_X(\tau_X)$, the Borel σ-algebra on X if there is no ambiguity on the topology τ_X. If (Ω, Σ) and (Φ, Γ) are two measurable spaces, then $\Sigma \otimes \Gamma$ denotes the smallest σ-algebra on $\Omega \times \Phi$ which contains all the sets $A \times B$, where $A \in \Sigma$, $B \in \Gamma$, i.e., $\Sigma \otimes \Gamma = \sigma_{\Omega \times \Phi}(\Sigma \times \Gamma)$. We note that the Borel σ-algebra $\mathcal{B}(X_1 \times X_2)$ contains $\mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$ in general. A map $f : \Omega \to \Phi$ is said to be (Σ, Γ) *measurable* (or simply, measurable) if for each $B \in \Gamma$, $f^{-1}(B) = \{x \in \Omega, f(x) \in B\} \in \Sigma$. Let X be a topological space and $F : (\Omega, \Sigma) \to 2^X$ be a map. F is said to be *measurable* (respectively, *weakly measurable*) if $F^{-1}(B) = \{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} \in \Sigma$ for each closed (respectively, open) subset B of X. The map F is said to have a *measurable* *graph* if $\operatorname{Graph} F := \{(\omega, y) \in \Omega \times X : y \in F(\omega)\} \in \Sigma \otimes \mathcal{B}(X)$. A function $f : \Omega \to X$ is a *measurable selection* of F if f is a measurable function such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

If (Ω, Σ) and (Φ, Γ) are measurable spaces, Y is a topological space, then a map $F : \Omega \times \Phi \to 2^Y$ is called (*jointly*) *measurable* (respectively, *weakly measurable*) if for every closed (respectively, open) subset B of Y, $F^{-1}(B) = \{(\omega, x) \in \Omega \times \Phi : F(\omega, x) \cap B \neq \emptyset\} \in \Sigma \otimes \Gamma$. In the case $\Phi = X$, a topological space, it is understood that Γ is the Borel σ-algebra $\mathcal{B}(X)$.

A topological space X is (i) a *Polish space* if F is separable and metrizable by a complete metric; (ii) a *Suslin* (respectively, *Polish*) *space* if X is a Hausdorff topological space and the continuous image of a Polish space. A *Suslin* (respectively, *Polish*) *subset* in a topological space is a subset which is a Suslin (respectively, Polish) space. Suslin sets play very important roles in measurable selection theory. We remark that if X_1 and X_2 are Suslin spaces, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$ (see e.g. [40, p. 113]).

Denote by $\mathcal{J}$ and $\mathcal{F}$ the sets of infinite and finite sequences of positive integers respectively. Let $\mathcal{G}$ be a family of sets and $F : \mathcal{F} \to \mathcal{G}$ be a map. For each $\sigma = (\sigma_i)_{i=1}^{\infty} \in \mathcal{J}$ and $n \in \mathbb{N}$, we shall denote $(\sigma_1, \ldots, \sigma_n)$ by $\sigma|n$; then $\bigcup_{\sigma \in \mathcal{J}} \bigcap_{n=1}^{\infty} F(\sigma|n)$ is said to be *obtained from $\mathcal{G}$ by the Suslin operation*. Now if every set obtained from $\mathcal{G}$ in this way is also in $\mathcal{G}$, then $\mathcal{G}$ is called a *Suslin family* (see e.g. [34], [41], [57] etc.).

Note that, if μ is an outer measure on a measurable space (Ω, Σ), then Σ is a Suslin family (see [41, p. 50]). In particular, if (Ω, Σ) is a complete measurable space, then Σ is a Suslin family (for more details, see [57, p. 864]). It also implies that the σ-algebra Σ of Lebesgue measurable subsets of [0,1] is a Suslin family.

Let X and Y be topological spaces, (Ω, Σ) be a measurable space and $F : \Omega \times X \to 2^Y$ be a map. Then (a) F is a *random operator* if for each fixed $x \in X$, the map $F(\cdot, x) : \Omega \to 2^Y$ is a measurable map; (b) F is *random lower semicontinuous* (respectively, *random upper semicontinuous*, *random continuous*) if F is a random operator and for each fixed $\omega \in \Omega$, $F(\omega, \cdot) : X \to 2^Y$ is lower semicontinuous (respectively, upper semicontinuous, continuous) and (c) a measurable (single-valued) map $\psi : \Omega \to X$ is said to be a *random maximal element* of the correspondence F if $F(\omega, \psi(\omega)) = 0$ for all $\omega \in \Omega$.

Let (Ω, Σ) be a measurable space, X be a topological space and $F : \Omega \times X \to 2^X$ be a map. The (single-valued) map $\varphi : \Omega \to X$ is said to be (i) a *deterministic fixed point* of F if $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for all $\omega \in \Omega$ and (ii) a *random fixed point* of F if φ is a measurable map and $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for all $\omega \in \Omega$. It should be noted here that some authors define a random fixed point of F to be a measurable map φ such that $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for almost every $\omega \in \Omega$ (see e.g. [38], [39] and the references therein).

Let I be any set of players and (Ω, Σ) be a measurable space. For each $i \in I$, let its strategy set X_i be a non-empty subset of a topological vector space. Let $X = \prod_{i \in I} X_i$. For each $i \in I$, let $P_i : \Omega \times X \to 2^{X_i}$ be a correspondence which is irreflexive, i.e. $x_i \notin P_i(\omega, x)$ for each $(\omega, x) \in \Omega \times X$. Following the terminology of Gale and Mas-Colell [21] in the deterministic case, the collection $\Gamma = (\Omega, X_i, P_i)_{i \in I}$ will be called a *random qualitative game*. A measurable map $\psi : \Omega \to X$ is said to be a *random equilibrium* of the random qualitative game Γ if $P_i(\omega, \psi(\omega)) = 0$ for all $i \in I$ and all $\omega \in \Omega$.

A *random generalized game* (or a *random abstract economy*) is a collection $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ where I is a (finite or infinite) set of players (agents) such that for each $i \in I$, X_i is a non-empty subset of a topological vector space and $A_i, B_i : \Omega \times X \to 2^{X_i}$ are random constraint correspondences where $X = \prod_{i \in I} X_i$, and $P_i : \Omega \times X \to 2^{X_i}$ is a preference correspondence (which are interpreted as for each player (or agent) $i \in I$, the associated constraint and preference correspondences A_i, B_i and P_i have stochastic actions). A *random equilibrium* of Γ is a (single-valued) measurable map $\Omega \to X$ such that for each $i \in I$, $\pi_i(\psi(\omega)) \in \overline{B}_i(\omega, \psi(\omega))$ and $A_i(\omega, \psi(\omega)) \cap P_i(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$. Here, π_i is the projection from X onto X_i. If $x \in X$, we shall also write x_i in place of $\pi_i(x)$ if there is no ambiguity. We remark that if A_i, B_i and P_i of the random generalized game $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ are independent of the variable $\omega \in \Omega$, i.e., $A_i(\omega, \cdot) = A_i(\cdot), B_i(\omega, \cdot) = B_i(\cdot)$ and $P_i(\omega, \cdot) = P_i(\cdot)$ for all $\omega \in \Omega$, and if $\overline{B}_i(\widehat{x}) = \mathrm{cl}_{X_i} B_i(\widehat{x})$ for each $\widehat{x} \in X$ (which is the case when B_i has a closed graph in $X \times X_i$; in particular, when $\mathrm{cl}\, B_i$ is upper semicontinuous with closed values), then our definition of an equilibrium point coincides with that of Ding *et al.* [16] in deterministic case; and if, in addition, $A_i = B_i$ for each $i \in I$, our definition of an equilibrium point coincides with the standard definition in deterministic case, e.g., in Borglin and Keiding [8], Tulcea [55] and Yannelis and Prabhakar [58].

We now recall two results which will be needed in this paper. The following is due to Leese [34, pp. 408–409]:

Theorem A. *Let (Ω, Σ) be a measurable space, Σ be a Suslin family, X be a Suslin space and $F : (\Omega, \Sigma) \to 2^X \setminus \{\emptyset\}$ be a mapping such that* $\mathrm{Graph}\, F \in \Sigma \otimes \mathcal{B}(X)$. *Then there exists a sequence $\{\phi_n : n = 1, 2, \dots\}$ of measurable selectors of F such that for each $\omega \in \Omega$, the set $\{\phi_n(\omega) : n = 1, 2, \dots\}$ is dense in $F(\omega)$.*

The following is Theorem 5.3 of Tan and Yuan [49]:

Theorem B. *Let $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

(a) *for each* $i \in I$, X_i *is a non-empty closed convex subset of a locally convex Hausdorff topological vector space* E_i *and* X_i *has property* (K);
(b) *for each* $i \in I$, B_i *is compact and upper semicontinuous with non-empty compact convex values and* $A_i(x) \subset B_i(x)$ *for each* $x \in X$;
(c) *for each* $i \in I$, P_i *is lower semicontinuous and* L*-majorized;*
(d) *for each* $i \in I$, $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ *is open in* X;
(e) *there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \setminus K$ *there is an* $x \in \mathrm{co}(X_0 \cup \{y\})$ *with* $x_i \in \mathrm{co}(A_i(y) \cap P_i(y))$ *for all* $i \in I$.

Then there exists $\overline{x} \in K$ *such that* $\overline{x}_i \in B_i(\overline{x})$ *and* $A_i(\overline{x}) \cap P_i(\overline{x}) = 0$ *for each* $i \in I$.

3. Random equilibria in locally convex spaces

By Theorems A and B, we have the following existence theorem for random equilibria of random generalized games:

THEOREM 3.1. *Let* (Ω, Σ) *be a measurable space,* Σ *be a Suslin family and* $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i\in I}$ *be a random generalized game such that* I *is countable and* $\mathrm{Dom}(A_i \cap P_i) \in \Sigma \otimes \mathcal{B}(X)$ *for each* $i \in I$. *Suppose that the following conditions are satisfied:*

(i) *for each* $i \in I$, X_i *is a non-empty convex Polish subset of a locally convex Hausdorff topological vector space* E_i;
(ii) *for each* $i \in I$ *and for each fixed* $\omega \in \Omega$, $B_i(\omega, \cdot)$ *is compact and upper semicontinuous with non-empty compact convex values, and for each* $(\omega, x) \in \Omega \times X$, $A_i(\omega, x) \subset B_i(\omega, x)$;
(iii) *for each* $i \in I$ *and for each fixed* $\omega \in \Omega$, $P_i(\omega, \cdot)$ *is lower semicontinuous and* L*-majorized;*
(iv) *for each* $i \in I$ *and* $\omega \in \Omega$, $E^i(\omega) = \{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ *is open in* X;
(v) *for each given* $\omega \in \Omega$, *there exist a non-empty compact subset* $K(\omega)$ *of* X *and a non-empty compact convex subset* $X_0(\omega)$ *of* X *such that for each* $y \in X \setminus K(\omega)$, *there is an* $x \in \mathrm{co}(X_0(\omega) \cup \{y\})$ *with* $x_i \in \mathrm{co}(A_i(\omega, y) \cap P_i(\omega, y))$ *for all* $i \in I$; *and*
(vi) *the mapping* $B : \Omega \times X \to 2^X$ *defined by* $B(\omega, x) = \prod_{i\in I} B_i(\omega, x)$ *for each* $(\omega, x) \in \Omega \times X$ *has a measurable graph, i.e.,* $\mathrm{Graph}\, B \in \Sigma \otimes (X \times X)$.

Then Γ *has a random equilibrium.*

PROOF. Define $\Psi : \Omega \to 2^{X\times X}$ by $\Psi(\omega) = \{(x, x) \in X \times X : A_i(\omega, x) \cap P_i(\omega, x) = \emptyset$ and $x \in B(\omega, x)$ for all $i \in I\}$ for each $\omega \in \Omega$. Then $\Psi(\omega) \neq \emptyset$ for each $\omega \in \Omega$ by assumptions (i)–(v) and Theorem B. Let $\Delta = \{(x, x) : x \in X\}$.

Then $\operatorname{Graph}\Psi = (((\Omega\times X)\setminus(\bigcup_{i\in I}(\operatorname{Dom}(A_i\cap P_i))))\times X)\cap(\operatorname{Graph}B)\cap(\Omega\times\Delta)$. By hypotheses, $\operatorname{Dom}(A_i \cap P_i) \in \Sigma\otimes\mathcal{B}(X_i)$ for each $i \in I$ and $\operatorname{Graph}B \in \Sigma\otimes\mathcal{B}(X\times X)$, so that $\bigcup_{i\in I}\operatorname{Dom}(A_i\cap P_i) \in \Sigma\otimes\mathcal{B}(X_i)$ since I is countable. Therefore $\operatorname{Graph}\Psi \in \Sigma\otimes\mathcal{B}(X\times X)$. Hence Ψ satisfies all conditions of Theorem A. By Theorem A, there exists a measurable selection ψ' of Ψ, where $\psi' : \Omega \to X\times X$. But then there exists $\psi : \Omega \to X$ such that $\psi'(\omega) = (\psi(\omega), \psi(\omega))$ for all $\omega \in \Omega$. Now if D is a closed subset of X, then $D\times D$ is a closed subset of $X\times X$; as $\psi^{-1}(D) = \{\omega\in\Omega : \psi(\omega) \in D\} = \{\omega \in \Omega : \psi'(\omega) \in D\times D\} \in \Sigma$, it follows that ψ is also measurable. Moreover, we have $A_i(\omega, \psi(\omega))\cap P_i(\omega, \psi(\omega)) = \emptyset$ and $\pi_i(\psi(\omega)) \in B_i(\omega, \psi(\omega))$ for all $\omega \in \Omega$ and all $i \in I$. $\square$

Remark. We note that if B_i has a measurable graph for each $i \in I$, then it is easy to see that the mapping $B : \Omega\times X \to 2^X$ defined by $B(\omega, x) = \prod_{i\in I} B_i(\omega, x)$ for each $(\omega, x) \in \Omega\times X$, by the same argument of Lemma 2.4 of Castaing [9, p. 96], also has a measurable graph. Thus, we have the following corollary:

Corollary 3.2. *Let (Ω,Σ) be a measurable space with Σ a Suslin family and $\mathcal{G} = (\Omega; X_i; A_i; P_i)_{i\in I}$ be a random abstract economy and let $X = \prod_{i\in I} X_i$. Suppose that I is countable and the following conditions are satisfied for each $i \in I$:*

- (a) *X_i is a non-empty compact convex Polish subset of a locally convex Hausdorff topological vector space E_i;*
- (b) *$A_i : \Omega\times X \to 2^{X_i}$ is such that for each fixed $\omega\in\Omega$, $A_i(\omega,\cdot)$ is upper semicontinuous with non-empty compact convex values and* $\operatorname{Graph}A_i \in \Sigma\otimes\mathcal{B}(X\times X_i)$*;*
- (c) *$P_i : \Omega\times X \to 2^{X_i}$ is such that for each fixed $\omega\in\Omega$, $P(\omega,\cdot)$ is lower semicontinuous and L-majorized and* $\operatorname{Dom}(A_i\cap P_i) \in \Sigma\otimes\mathcal{B}(X)$*;*
- (d) *for each $\omega\in\Omega$, $E^i(\omega) = \{x \in X : A_i(\omega, x)\cap P_i(\omega, x) \neq \emptyset\}$ is open in X.*

Then $\mathcal{G}$ has a random equilibrium.

By taking $A_i = B_i = X_i$ for all $i \in I$ in Theorem 3.1 and noting that the domain of a lower semicontinuous correspondence is open, we have the following existence theorem for a random qualitative game:

Theorem 3.3. *Let (Ω,Σ) be a measurable space, Σ be a Suslin family and $\Gamma = (\Omega, X_i, P_i)_{i\in I}$ be a random qualitative game such that I is a countable and* $\operatorname{Dom}P_i \in \Sigma\otimes\mathcal{B}(X)$ *for each $i \in I$. Suppose that the following conditions are satisfied:*

- (i) *for each $i \in I$, X_i is a non-empty compact convex Polish subset of a locally convex Hausdorff topological vector space E_i;*

(ii) *for each* $i \in I$ *and for each fixed* $\omega \in \Omega$, $P_i(\omega, \cdot\,)$ *is lower semicontinuous and* L*-majorized.*

Then Γ *has a random equilibrium.*

4. Non-compact random quasi-variational inequalities

In this section, by our existence theorem for random equilibria of random generalized games, namely, Theorem 3.1, some existence theorems for random quasi-variational inequalities and generalized random quasi-variational inequalities are given. Our results not only generalize the results of Tan [50] and Zhang [59], but also they are the stochastic versions of corresponding results in the literature (see e.g. [4]–[5], [14], [46], [59]–[60] and the references therein).

Here we emphasize that our arguments for the existence of solutions for random quasi-variational inequalities are different from the approaches used in [50] and [59].

Theorem 4.1. *Let* (Ω, Σ) *be a measurable space with* Σ *a Suslin family, and let* I *be countable. For each* $i \in I$, *suppose that the following conditions are satisfied:*

(a) X_i *is a non-empty convex Polish subset of a locally convex Hausdorff topological vector space;*

(b) *for each fixed* $\omega \in \Omega$, $A_i(\omega, \cdot\,) : X = \prod_{i\in I} X_i \to 2^{X_i}$ *is compact and upper semicontinuous with non-empty compact convex values, and* $\operatorname{Graph} A_i \in \Sigma \otimes \mathcal{B}(X \times X_i)$;

(c) $\psi_i : \Omega \times X \times X_i \to \mathbb{R} \cup \{-\infty, +\infty\}$ *is such that:*

(c)$_1$ $x \mapsto \psi_i(\omega, x, y)$ *is lower semicontinuous on* X *for each fixed* $(\omega, y) \in \Omega \times X_i$;

(c)$_2$ $x_i \notin \operatorname{co}\{y \in X_i : \psi_i(\omega, x, y) > 0\}$ *for each fixed* $(\omega, x) \in \Omega \times X$;

(c)$_3$ *for each fixed* $\omega \in \Omega$, *the set* $\{x \in X : \alpha_i(\omega, x) > 0\}$ *is open in* X, *where* $\alpha_i : \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ *is defined by* $\alpha_i(\omega, x) = \sup_{y_i \in A_i(\omega,x)} \psi_i(\omega, x, y_i)$ *for each* $(\omega, x) \in \Omega \times X$;

(d) $\{(\omega, x) \in \Omega \times X : \alpha_i(\omega, x) > 0\} \in \Sigma \otimes \mathcal{B}(X)$;

(e) *for each given* $\omega \in \Omega$, *there exist a non-empty compact convex subset* $X_0(\omega)$ *of* X *and a non-empty compact subset* $K(\omega)$ *of* X *such that for each* $y \in X \setminus K(\omega)$ *there exists* $x \in \operatorname{co}(X_0(\omega) \cup \{y\})$ *with* $x_i \in \operatorname{co}(A_i(\omega, y) \cap \{z \in X_i : \psi_i(\omega, y, z) > 0\})$.

Then there exists a measurable map $\phi : \Omega \to X$ *such that for each* $i \in I$, $\pi_i(\phi(\omega)) \in A_i(\omega, \phi(\omega))$ *and*

$$\sup_{y \in A_i(\omega,\phi(\omega))} \psi_i(\omega, \phi(\omega), y) \le 0$$

for all $\omega \in \Omega$.

PROOF. For each $i \in I$, define $P_i : \Omega \times X \to 2^{X_i}$ by setting $P_i(\omega, x) = \{y \in X_i : \psi_i(\omega, x, y) > 0\}$ for each $(\omega, x) \in \Omega \times X$. We shall show that $G = (\Omega; X_i; A_i; P_i)_{i \in I}$ satisfies all hypotheses of Theorem 3.1 with $A_i = B_i$ for all $i \in I$.

Suppose $i \in I$ and $\omega \in \Omega$. By $(c)_1$, for each fixed $y \in X_i$, $(P_i(\omega, \cdot))^{-1}(y) = \{x \in X : \psi_i(\omega, x, y) > 0\}$ is open in X and by $(c)_2$, $x_i \notin \operatorname{co} P_i(\omega, x)$ for each $x \in X$. This shows that $P_i(\omega, \cdot)$ is lower semicontinuous and is of class L and hence is L-majorized. By the definition of α_i, we note that $\{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\} = \{x \in X : \alpha_i(\omega, x) > 0\}$ so that $\{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is open in X by $(c)_3$. By (d), we have $\operatorname{Dom}(A_i \cap P_i) \in \Sigma \otimes \mathcal{B}(X)$. By (b), $\operatorname{Graph} A_i \in \Sigma \otimes \mathcal{B}(X \times X)$, so that the mapping $A : \Omega \times X \to X$ defined by $A(\omega, x) = \prod_{i \in I} A_i(\omega, x)$ for each $(\omega, x) \in \Omega \times X$ has a measurable graph (see e.g. the argument of Castaing [9, p. 96]). Therefore $G = (\Omega; X_i; A_i; P_i)_{i \in I}$ satisfies all hypothesis of Theorem 3.1 with $A_i = B_i$ for each $i \in I$. By Theorem 3.1, there exists a measurable map $\phi : \Omega \to X$ such that for each $i \in I$, $\pi_i(\phi(\omega)) \in A_i(\omega, \phi(\omega))$ and $A_i(\omega, \phi(\omega)) \cap P_i(\omega, \phi(\omega)) = \emptyset$ for all $\omega \in \Omega$, i.e., $\pi_i(\phi(\omega)) \in A_i(\omega, \phi(\omega))$ and $\sup_{y \in A_i(\omega, \phi(\omega))} \psi_i(\omega, \phi(\omega), y) \leq 0$ for all $\omega \in \Omega$. $\square$

By letting $I = \{1\}$ in Theorem 4.1, we have the following existence result for random quasi-variational inequalities:

THEOREM 4.2. *Let (Ω, Σ) be a measurable space with Σ a Suslin family. Suppose that the following conditions are satisfied:*

(a) *X is a non-empty convex Polish subset of a locally convex Hausdorff topological vector space;*
(b) *for each fixed $\omega \in \Omega$, $A(\omega, \cdot) : X \to 2^X$ is compact and upper semicontinuous with non-empty compact and convex values, and* $\operatorname{Graph} A \in \Sigma \otimes \mathcal{B}(X \times X)$*;*
(c) *$\psi : \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that:*
   $(c)_1$ *$x \mapsto \psi(\omega, x, y)$ is lower semicontinuous on X for each fixed $(\omega, y) \in \Omega \times X$;*
   $(c)_2$ *$x \notin \operatorname{co}\{y \in X : \psi(\omega, x, y) > 0\}$ for each fixed $(\omega, x) \in \Omega \times X$;*
   $(c)_3$ *for each fixed $\omega \in \Omega$, the set $\{x \in X : \alpha(\omega, x) > 0\}$ is open in X, where $\alpha : \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $\alpha(\omega, x) = \sup_{y \in A(\omega, x)} \psi(\omega, x, y)$ for each $(\omega, x) \in \Omega \times X$;*
(d) *$\{(\omega, x) \in \Omega \times X : \alpha(\omega, x) > 0\} \in \Sigma \otimes \mathcal{B}(X)$;*
(e) *for each given $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \setminus K(\omega)$ there exists $x \in \operatorname{co}(X_0(\omega) \cup \{y\})$ with $x \in \operatorname{co}(A(\omega, y) \cap \{z \in X : \psi(\omega, y, z) > 0\})$.*

Then there exists a measurable map $\phi : \Omega \to X$ *such that* $\phi(\omega) \in A(\omega, \phi(\omega))$ *and*

$$\sup_{y \in A(\omega,\phi(\omega))} \psi(\omega, \phi(\omega), y) \le 0$$

for all $\omega \in \Omega$.

As a consequence of Theorem 4.2 and Theorem 4.2(c) of [57], we have the following random fixed point theorem:

Corollary 4.3. *Let* (Ω, Σ) *be a measurable space with* Σ *a Suslin family,* X *be a non-empty compact convex Polish subset of a locally convex Hausdorff topological vector space and* $A : \Omega \times X \to 2^X$ *be measurable. If for each* $\omega \in \Omega$, $A(\omega, \cdot\,)$ *is upper semicontinuous with non-empty compact convex values, then* A *has a random fixed point.*

For more details about random fixed point theorems, we refer the reader to [47] and the references therein.

5. Generalized random quasi-variational inequalities

In this section, by applying results in Section 4, we shall consider the following generalized random variational inequality problems $(*)$ and $(**)$.

Let (Ω, Σ) be a measurable space, X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space E and E^* be the dual space of E. Suppose the correspondences $F : \Omega \times X \to 2^X$, $T : \Omega \times X \to 2^{E^*}$ and the function $f : \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ are given. We want to find a measurable map $\psi : \Omega \to X$ which satisfies the following generalized random quasi-variational inequalities:

$$(*) \qquad \begin{cases} \psi(\omega) \in F(\omega, \psi(\omega)) \\ \sup\limits_{y \in F(\omega,\psi(\omega))} [\sup\limits_{u \in T(\omega,\psi(\omega))} \mathrm{Re}\langle u, \psi(\omega) - y\rangle + f(\omega, \psi(\omega), y)] \le 0 \end{cases}$$

for all $\omega \in \Omega$. We also want to find two measurable maps $\psi : \Omega \to X$ and $\phi : \Omega \to E^*$ such that

$$(**) \qquad \begin{cases} \psi(\omega) \in F(\omega, \psi(\omega)) \quad \text{and} \quad \phi(\omega) \in T(\omega, \psi(\omega)), \\ \mathrm{Re}\langle \phi(\omega), \psi(\omega) - y\rangle + f(\omega, \psi(\omega), y) \le 0 \end{cases}$$

for all $y \in F(\omega, \psi(\omega))$ and for all $\omega \in \Omega$.

Now we recall some definitions (see e.g. [60]). Let X be a convex subset of a topological vector space E. A function $\psi(x, y) : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be

(1) γ-*diagonally quasi-convex* (respectively, γ-*diagonally quasi-concave*) in y, for short, γ-DQCX (respectively, γ-DQCV) in y, if for each $A \in \mathcal{F}(X)$ and each $y \in \mathrm{co}(A)$, $\gamma \le \max_{x \in A} \psi(y, x)$ (respectively, $\gamma \ge \inf_{x \in A} \psi(y, x)$);

(2) γ-*diagonally convex* (respectively, γ-*diagonally concave*) in y, for short, γ-DCX (respectively, γ-DCV) in y, if for each $A \in \mathcal{F}(X)$ and each $y \in \mathrm{co}(A)$ with $y = \sum_{i=1}^m \lambda_i y_i$ ($\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$), we have $\gamma \leq \sum_{i=1}^m \lambda_i \psi(y, y_i)$ (respectively, $\gamma \geq \sum_{i=1}^m \lambda_i \psi(y, y_i)$).

Let X and Y be two non-empty convex subsets of E. We also recall that a function $\psi : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *quasi-convex* (respectively, *quasi-concave*) in y if for each fixed $x \in X$, for each $A \in \mathcal{F}(Y)$ and each $y \in \mathrm{co}(A)$, $\psi(x, y) \leq \max_{z \in A} \psi(x, z)$ (respectively, $\psi(x, y) \geq \min_{z \in A} \psi(x, z)$).

It is easy to see that (i) if $\psi(x, y)$ is γ-DCX (respectively, γ-DCV) in y, then $\psi(x, y)$ is γ-DQCX (respectively, γ-DQCV) in y and (ii) if $\psi_i : X \times Y \to \mathbb{R}$ is γ-DCX (respectively, γ-DCV) in y for each $i = 1, 2, \ldots, m$, then $\psi(x, y) = \sum_{i=1}^m a_i(x) \psi_i(x, y)$ is also γ-DCX (respectively, γ-DCV) in y, where $a_i : X \to \mathbb{R}$ with $a_i(x) \geq 0$ and $\sum_{i=1}^m a_i(x) = 1$ for each $x \in X$ and (iii) the function $\psi(x, y) : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is 0-DQCV in y if and only if $x \notin \mathrm{co}(\{y \in X : \psi(x, y) > 0\})$ for each $x \in X$.

First we consider the following existence theorem for solutions of problem $(*)$ for which monotonicity is needed.

THEOREM 5.1. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X be a non-empty convex Polish subset of a locally convex Hausdorff topological vector space E. Suppose that the following conditions are satisfied:*

(i) *$F : \Omega \times X \to 2^X$ is such that for each fixed $\omega \in \Omega$, $F(\omega, \cdot)$ is compact and upper semicontinuous with non-empty compact convex values, and* $\mathrm{Graph}\, F \in \Sigma \otimes \mathcal{B}(X \times X)$*;*

(ii) *$T : \Omega \times X \to 2^{E^*}$ is such that for each fixed $\omega \in \Omega$, $T(\omega, \cdot)$ is monotone (i.e.,* $\mathrm{Re}\langle u - v, y - x\rangle \geq 0$ *for all $u \in T(\omega, y)$ and $v \in T(\omega, x)$ and for all $x, y \in X$) with non-empty values and for each one-dimensional flat $L \subset E, T(\omega, \cdot)|_{L \cap X}$ is lower semicontinuous from the relative topology of X into the weak*-topology $\sigma(E^*, E)$ of E^*;*

(iii) *$f : \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X and for each fixed $(\omega, x) \in \Omega \times X, y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$ for each $(\omega, x) \in \Omega \times X$;*

(iv) *for each fixed $\omega \in \Omega$, the set*

$$\{x \in X : \sup_{y \in F(\omega, x)} [\sup_{u \in T(\omega, y)} \mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\}$$

is open in X;

(v) *the set $\{(\omega, x) \in \Omega \times X : \sup_{y \in F(\omega, x)} [\sup_{u \in T(\omega, y)} \mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathcal{B}(X)$;*

(vi) *for each given* $\omega \in \Omega$, *there exist a non-empty compact convex subset* $X_0(\omega)$ *of* X *and a non-empty compact subset* $K(\omega)$ *of* X *such that for each* $x \in X \setminus K(\omega)$ *there exists* $y \in \mathrm{co}(X_0(\omega) \cup \{x\})$ *with* $y \in \mathrm{co}(F(\omega, x) \cap \{z \in X : \sup_{u \in T(\omega,z)} \mathrm{Re}\langle u, x - z\rangle + f(\omega, x, z) > 0\})$.

Then there exists a measurable map $\phi : \Omega \to X$ *such that* $\phi(\omega) \in F(\omega, \phi(\omega))$ *and*

$$\sup_{u \in T(\omega,\phi(\omega))} \mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y) \le 0$$

for all $y \in F(\omega, \phi(\omega))$ *and* $\omega \in \Omega$.

Proof. Define a function $\psi : \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(\omega, x, y) = \sup_{u \in T(\omega,y)} \mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Then by (iii), $x \mapsto \psi(\omega, x, y)$ is lower semicontinuous on X for each $(\omega, y) \in \Omega \times X$. For each $\omega \in \Omega$, since $T(\omega, \cdot\,)$ is monotone, by (iii), it is easy to prove that $\psi(\omega, x, y)$ is 0-DCV in y by Proposition 3.2 of Zhou and Chen [60]. The conditions (i)–(vi) imply that all hypotheses of Theorem 4.2 are satisfied. By Theorem 4.2, there exists a measurable map $\phi : \Omega \to X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega,\phi(\omega))} \sup_{u \in T(\omega,y)} [\mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \le 0 \tag{1}$$

for all $\omega \in \Omega$.

We shall now modify the proof of Theorem 3 of Tan [46] to prove that

$$\sup_{y \in F(\omega,\phi(\omega))} \sup_{u \in T(\omega,\phi(\omega))} [\mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \le 0$$

for each $\omega \in \Omega$.

Fix an $\omega \in \Omega$. Let $x \in F(\omega, \phi(\omega))$ be arbitrarily given and let

$$z_t(\omega) = tx + (1 - t)\phi(\omega) = \phi(\omega) - t(\phi(\omega) - x)$$

for $t \in [0, 1]$. As $F(\omega, \phi(\omega))$ is convex, we have $z_t(\omega) \in F(\omega, \phi(\omega))$ for $t \in [0, 1]$. Therefore by (1) we have

$$\sup_{u \in T(\omega,z_t(\omega))} [\mathrm{Re}\langle u, \phi(\omega) - z_t(\omega)\rangle + f(\omega, \phi(\omega), z_t(\omega))] \le 0$$

for all $t \in [0, 1]$.

Since for each $x \in X$, $y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$, it follows that for $t \in (0, 1]$,

$$\begin{aligned} t \cdot \Big\{ &\sup_{u \in T(\omega, z_t(\omega))} [\mathrm{Re}\langle u, \phi(\omega) - x\rangle] + f(\omega, \phi(\omega), x)\Big\} \\ &\le \sup_{u \in T(\omega, z_t(\omega))} t \cdot [\mathrm{Re}\langle u, \phi(\omega) - x\rangle] + f(\omega, \phi(\omega), tx + (1-t)\phi(\omega)) \\ &= \sup_{u \in T(\omega, z_t(\omega))} \mathrm{Re}\langle u, \phi(\omega) - z_t(\omega)\rangle + f(\omega, \phi(\omega), z_t(\omega)) \le 0, \end{aligned}$$

which implies that for $t \in (0, 1]$,

$$\sup_{u \in T(\omega, z_t(\omega))} \mathrm{Re}\langle u, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x) \le 0. \tag{2}$$

Let $z_0 \in T(\omega, \phi(\omega))$ be arbitrarily fixed. For each $\varepsilon > 0$, let

$$U_{z_0} = \{z \in E^* : |\mathrm{Re}\langle z_0 - z, \phi(\omega) - x\rangle| < \varepsilon\}.$$

Then U_{z_0} is a $\sigma(E^*, E)$-neighborhood of z_0. Since $T(\omega, \cdot)|_{L \cap X}$ is lower semicontinuous, where $L := \{z_t(\omega) : T \in [0, 1]\}$, and $U_{z_0} \cap T(\omega, \phi(\omega)) \neq \emptyset$, there exists a neighborhood $N(\phi(\omega))$ of $\phi(\omega)$ in L such that if $z \in N(\phi(\omega))$, then $T(\omega, \phi(\omega)) \cap U_z \neq \emptyset$. But then there exists $\delta \in (0, 1]$ such that $z_t(\omega) \in N(\phi(\omega))$ for all $t \in (0, \delta)$. Fix any $t \in (0, \delta)$ and $u \in T(\omega, z_t(\omega)) \cap U_{z_0}$. We have $|\mathrm{Re}\langle z_0 - u, \phi(\omega) - x\rangle| < \varepsilon$. This implies that $\mathrm{Re}\langle z_0, \phi(\omega) - x\rangle < \mathrm{Re}\langle u, \phi(\omega) - x\rangle + \varepsilon$. Therefore $\mathrm{Re}\langle z_0, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x) < \mathrm{Re}\langle u, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x) + \varepsilon < \varepsilon$ by (2). Since $\varepsilon > 0$ is arbitrary, $\mathrm{Re}\langle z_0, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x) \le 0$. As $z_0 \in T(\omega, \phi(\omega))$ is arbitrary,

$$\sup_{z \in T(\omega, \phi(\omega))} \mathrm{Re}\langle z, \phi(\omega) - x\rangle + f(\omega, \phi(\omega), x) \le 0$$

for all $x \in F(\omega, \phi(\omega))$. □

THEOREM 5.2. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X be a non-empty bounded convex Polish subset of a locally convex Hausdorff topological vector space E. Assume that $F : \Omega \times X \to 2^X$ is such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact and convex values and* $\mathrm{Graph}\, F \in \Sigma \otimes \mathcal{B}(X \times X)$, *and $T : \Omega \times X \to 2^{E^*}$ is such that for each given $\omega \in \Omega$, $T(\omega, \cdot)$ is monotone with non-empty values and is lower semicontinuous from the relative topology of X to the strong topology of E^*. Suppose that*

(i) *$f : \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that for each given $\omega \in \Omega$, $(x, y) \mapsto f(\omega, x, y)$ is lower semicontinuous and for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$ for each $(\omega, x) \in \Omega \times X$;*

(ii) *the set $\{(\omega, x) \in \Omega \times X : \sup_{y \in F(\omega, x)} \sup_{u \in T(\omega, y)} [\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathcal{B}(X)$;*

(iii) *for each* $\omega \in \Omega$, *there exist a non-empty compact convex subset* $X_0(\omega)$ *of* X *and a non-empty compact subset* $K(\omega)$ *of* X *such that for each* $x \in X \setminus K(\omega)$ *there exists* $y \in \mathrm{co}(X_0(\omega) \cup \{x\})$ *with* $y \in \mathrm{co}(F(\omega, x) \cap \{z \in X : \sup_{u \in T(\omega,z)} \mathrm{Re}\langle u, x - z\rangle + f(\omega, x, z) > 0\})$.

Then there exists a measurable map $\phi : \Omega \to X$ *such that* $\phi(\omega) \in F(\omega, \phi(\omega))$ *and*

$$\sup_{y \in F(\omega,\phi(\omega))} \Big[\sup_{u \in T(\omega,\phi(\omega))} \mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)\Big] \le 0$$

for all $\omega \in \Omega$.

Proof. By Theorem 5.1, we need only show that for each given $\omega \in \Omega$, the set

$$\Sigma(\omega) := \{x \in X : \sup_{y \in F(\omega,x)} \Big[\sup_{u \in T(\omega,y)} \mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)\Big] > 0\}$$

is open in X.

Since X is bounded and $f(\omega, \cdot, \cdot)$ is lower semicontinuous, the function $(u, x, y) \mapsto \mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)$ is lower semicontinuous from $E^* \times X \times X$ to $\mathbb{R}$ for each fixed $\omega \in \Omega$. Therefore $(x, y) \mapsto \sup_{u \in T(\omega,y)}[\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)]$ is also lower semicontinuous by lower semicontinuity of $T(\omega, \cdot)$ and Proposition III-19 of Aubin and Ekeland [5, p. 118]. Since $F(\omega, \cdot)$ is lower semicontinuous, $x \mapsto \sup_{y \in F(\omega,x)} \sup_{u \in T(\omega,y)}[\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)]$ is again lower semicontinuous for each fixed $\omega \in \Omega$. Thus the set

$$\Sigma(\omega) = \{x \in X : \sup_{y \in F(\omega,x)} \sup_{u \in T(\omega,y)} [\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\}$$

is open in X. □

Now we will consider the existence of solutions for the problems $(*)$ and $(**)$ without assuming monotonicity as in Theorems 5.1 and 5.2.

Theorem 5.3. *Let* (Ω, Σ) *be a measurable space with* Σ *a Suslin family and* X *be a non-empty convex Polish subset of a locally convex Hausdorff topological vector space* E. *Suppose that:*

(i) $F : \Omega \times X \to 2^X$ *is such that for each* $\omega \in \Omega$, $F(\omega, \cdot)$ *is compact and upper semicontinuous with non-empty compact convex values, and* $\mathrm{Graph}\, F \in \Sigma \otimes \mathcal{B}(X \times X)$;

(ii) $T : \Omega \times X \to 2^{E^*}$ *is such that* $x \mapsto \inf_{u \in T(\omega,x)} \mathrm{Re}\langle u, x - y\rangle$ *is lower semicontinuous for each* $(\omega, y) \in \Omega \times X$;

(iii) $f : \Omega \times X \times X \to \mathbb{R}$ *is such that for each fixed* $(\omega, y) \in \Omega \times X$, $x \mapsto f(\omega, x, y)$ *is lower semicontinuous on* X *and for each fixed* $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ *is* 0*-diagonally concave;*

(iv) *for each given* $\omega \in \Omega$, *the set*

$$\{x \in X : \sup_{y \in F(\omega,x)} [\inf_{u \in T(\omega,x)} \mathrm{Re}\langle u, x-y\rangle + f(\omega,x,y)] > 0\}$$

is open in X*;*

(v) *the set* $\{(\omega,x) \in \Omega \times X : \sup_{y \in F(\omega,x)} \inf_{u \in T(\omega,x)} [\mathrm{Re}\langle u, x-y\rangle + f(\omega,x,y)] > 0\} \in \Sigma \otimes \mathcal{B}(X)$*;*

(vi) *for each* $\omega \in \Omega$, *there exist a non-empty compact convex subset* $X_0(\omega)$ *of* X *and a non-empty compact subset* $K(\omega)$ *of* X *such that for each* $x \in X \setminus K(\omega)$ *there exists* $y \in \mathrm{co}(X_0(\omega) \cup \{x\})$ *with* $y \in \mathrm{co}(F(\omega,x) \cap \{z \in X : \sup_{u \in T(\omega,z)} \mathrm{Re}\langle u, x-z\rangle + f(\omega,x,z) > 0\})$.

Then there exists a measurable map $\phi : \Omega \to X$ *such that* $\phi(\omega) \in F(\omega,\phi(\omega))$ *and*

$$\inf_{u \in T(\omega,\phi(\omega))} \mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega,\phi(\omega),y) \le 0$$

for all $y \in F(\omega,\phi(\omega))$ *and* $\omega \in \Omega$.

Suppose that, in addition, (1) *for each fixed* $(\omega,x) \in \Omega \times X$, $y \mapsto f(\omega,x,y)$ *is lower semicontinuous and concave and* f *is measurable;* (2) *there exists a non-empty Polish subset* E_0^* *of* E^* *such that* $T(\Omega \times X) \subset E_0^*$, T *is measurable with non-empty strongly compact convex values and* (3) F *is measurable.*

Then there exists a measurable function $\rho : \Omega \to E^*$ *such that* $\rho(\omega) \in T(\omega,\phi(\omega))$ *and*

$$\sup_{y \in F(\omega,\phi(\omega))} \{\mathrm{Re}\langle \rho(\omega), \phi(\omega) - y\rangle + f(\omega,\phi(\omega),y)\} \le 0$$

for all $\omega \in \Omega$.

PROOF. Define $\psi : \Omega \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(\omega,x,y) = \inf_{u \in T(\omega,x)} \mathrm{Re}\langle u, x-y\rangle + f(\omega,x,y),$$

for each $(\omega,x,y) \in \Omega \times X \times X$. Then by (ii), (iii) and (iv) we have:

(1) for each fixed $(\omega,y) \in \Omega \times X$, $x \mapsto \psi(\omega,x,y)$ is lower semicontinuous on X and $x \notin \mathrm{co}(\{y \in X : \psi(\omega,x,y) > 0\})$ for each $(\omega,x) \in \Omega \times X$;

(2) for each $\omega \in \Omega$, the set $\{x \in X : \sup_{y \in F(\omega,x)} \psi(\omega,x,y) > 0\}$ is open in X.

Therefore F and ψ satisfy all conditions of Theorem 4.2. By Theorem 4.2 there exists a measurable map $\phi : \Omega \to X$ such that $\phi(\omega) \in F(\omega,\phi(\omega))$ and

$$\sup_{y \in F(\omega,\phi(\omega))} \inf_{u \in T(\omega,\phi(\omega))} [\mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega,\phi(\omega),y)] \le 0$$

for all $\omega \in \Omega$.

If, in addition, the conditions (1), (2) and (3) hold, we shall find another measurable map $\rho : \Omega \to E^*$ such that $\rho(\omega) \in T(\omega, \phi(\omega))$ and $\sup_{y \in F(\omega,\phi(\omega))}[\mathrm{Re}\langle \rho(\omega), \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \le 0$ for each $\omega \in \Omega$.

Fix an $\omega \in \Omega$. Define $f_1 : F(\omega, \phi(\omega)) \times T(\omega, \phi(\omega)) \to \mathbb{R}$ by

$$f_1(y, u) = \mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)$$

for each $(y, u) \in F(\omega, \phi(\omega)) \times T(\omega, \phi(\omega))$. Then for each $y \in F(\omega, \phi(\omega))$, $u \mapsto f_1(y, u)$ is lower semicontinuous and convex and for each fixed $u \in T(\omega, \phi(\omega))$, $y \mapsto f_1(y, u)$ is concave. By Kneser's minimax theorem [33],

$$\begin{aligned}\inf_{u \in T(\omega,\phi(\omega))} \sup_{y \in F(\omega,\phi(\omega))} &[\mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \\ &= \sup_{y \in F(\omega,\phi(\omega))} \inf_{u \in T(\omega,\phi(\omega))} [\mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \le 0.\end{aligned}$$

Since $T(\omega, \phi(\omega))$ is compact, there exists $u_0 \in T(\omega, \phi(\omega))$ such that

$$\sup_{y \in F(\omega,\phi(\omega))} [\mathrm{Re}\langle u_0, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \le 0.$$

Now let $\Phi, T_1 : \Omega \to 2^X$ be defined by

$$\begin{aligned}\Phi(\omega) &= \{u \in T(\omega, \phi(\omega)) : \sup_{y \in F(\omega,\phi(\omega))} [\mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \le 0\}, \\ T_1(\omega) &= T(\omega, \phi(\omega))\end{aligned}$$

for each $\omega \in \Omega$. Note that $\Phi(\omega) \ne \emptyset$ for all $\omega \in \Omega$. Since T and ϕ are measurable, T_1 is also measurable by Lemma 3 in [39, p. 55].

Define $g_1 : \Omega \times X \times X \times E_0^* \to \mathbb{R}$ by

$$g_1(\omega, x, y, u) = \mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)$$

for each $(\omega, x, y, u) \in \Omega \times X \times X \times E_0^*$. Then g_1 is measurable. Also define $g_2 : \Omega \times X \times E_0^* \to \mathbb{R}$ by

$$g_2(\omega, y, u) = \mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)$$

for each $(\omega, y, u) \in \Omega \times X \times E_0^*$. Now define $F_1 : \Omega \to 2^X$ by $F_1(\omega) = F(\omega, \phi(\omega))$ for each $\omega \in \Omega$. Since ϕ is measurable and F is also measurable, g_2 and F_1 are measurable by Lemma 3 in [39, p. 55] again.

Now define $g_3 : \Omega \times E_0^* \to \mathbb{R}$ by

$$g_3(\omega, u) = \sup_{y \in F(\omega,\phi(\omega))} g_2(\omega, y, u) = \sup_{y \in F(\omega,\phi(\omega))} [\mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)]$$

for each $(\omega, u) \in \Omega \times E_0^*$. We shall prove that g_3 is a measurable function.

Since F_1 is measurable, by Theorem A, there exists a countable family of measurable maps $p_n : \Omega \to X$ such that $F_1(\omega) = \mathrm{cl}\{p_n(\omega) : n = 1, 2, \ldots\}$ for each $\omega \in \Omega$. Since ϕ is measurable, for each fixed $(u, y) \in E^* \times X$, the map

$\omega \mapsto \mathrm{Re}\langle u, \phi(\omega) - y\rangle$ is measurable. Note that the map $(u, y) \mapsto \mathrm{Re}\langle u, \phi(\omega) - y\rangle$ is continuous, so that the map $(\omega, u, y) \mapsto \mathrm{Re}\langle u, \phi(\omega) - y\rangle$ is measurable by Theorem III.14 of Castaing and Valadier [10, p. 70]. For each $n \in \mathbb{N}$, the function $g_n' : \Omega \times E^* \to \mathbb{R}$ defined by

$$g_n'(\omega, u) = \mathrm{Re}\langle u, \phi(\omega) - p_n(\omega)\rangle + f(\omega, \phi(\omega), p_n(\omega))$$

for each $(\omega, u) \in \Omega \times E^*$, is measurable. Therefore for each $n \in \mathbb{N}$, the map $(\omega, u) \mapsto \mathrm{Re}\langle u, \phi(\omega) - p_n(\omega)\rangle + f(\omega, \phi(\omega), p_n(\omega))$ is also measurable. Since for each $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is lower semicontinuous, it follows that for each $r \in \mathbb{R}$,

$$\begin{aligned}\{(\omega, u) \in \Omega \times E^* &: g_3(\omega, u) \le r\} \\ &= \bigcap_{n=1}^{\infty} \{(\omega, u) \in \Omega \times E^* : g_n'(\omega, u) \le r\} \in \Sigma \otimes \mathcal{B}(E^*).\end{aligned}$$

Therefore the function g_3 is measurable so that the set $M_0 = \{(\omega, u) \in \Omega \times E_0^* : g_3(\omega, u) \le 0\} \in \Sigma \otimes \mathcal{B}(E^*)$. Hence $\mathrm{Graph}\,\Phi = (\mathrm{Graph}\,T_1) \cap M_0 \in \Sigma \otimes \mathcal{B}(E_0^*)$. By Theorem A, there exists a measurable map $\rho : \Omega \to E_0^*$ such that $\rho(\omega) \in \Phi(\omega)$ for each $\omega \in \Omega$. By the definition of Φ, the measurable map ρ satisfies the following:

$$(**) \qquad \begin{cases} \phi(\omega) \in F(\omega, \phi(\omega)) \quad \text{and} \quad \rho(\omega) \in T(\omega, \phi(\omega)), \\ \sup_{y \in F(\omega, \phi(\omega))} [\mathrm{Re}\langle \rho(\omega), \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)] \le 0. \end{cases}$$

□

Note that if X is bounded and the mapping $T : \Omega \times X \to 2^{E^*}$ is such that for each $\omega \in \Omega$, $T(\omega, \cdot\,)$ is upper semicontinuous with non-empty strongly compact values, then by Lemma 2 of Kim and Tan in [32, p. 140] or Theorem 1 of Aubin in [4, p. 67], the condition (ii) of Theorem 5.3 is satisfied. Thus Theorem 5.3 is a stochastic version of Theorem 3 of Shih and Tan in [44, p. 340]. Recall that for a topological vector space E, the strong topology on its dual space E^* is generated by the family $\{U(B; \varepsilon) : B$ is a non-empty bounded subset of E and $\varepsilon > 0\}$ as a base for the neighborhood system at zero, where $U(B; \varepsilon) := \{f \in E^* : \sup_{x \in B} |\mathrm{Re}\langle f, x\rangle| < \varepsilon\}$.

Now if we impose the upper semicontinuity condition on the correspondence T, then we have the following:

Theorem 5.4. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X be a non-empty bounded convex Polish subset of a locally convex Hausdorff topological vector space E. Suppose that*

(i) *$F : \Omega \times X \to 2^X$ is random compact and continuous with non-empty compact convex values;*

(ii) *$T : \Omega \times X \to 2^{E^*}$ is such that for each given $\omega \in \Omega$, $T(\omega, \cdot)$ is compact and upper semicontinuous with non-empty strongly compact and convex values;*

(iii) *$f : \Omega \times X \times X \to \mathbb{R}$ is such that* (a) *for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X;* (b) *for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is 0-diagonally concave;*

(iv) *the set $\{(\omega, x) \in \Omega \times X : \sup_{y \in F(\omega,x)} \inf_{u \in T(\omega,x)} [\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathcal{B}(X)$.*

(v) *for each $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \setminus K(\omega)$ there exists $y \in \mathrm{co}(X_0(\omega) \cup \{x\})$ with $y \in \mathrm{co}(F(\omega, x) \cap \{z \in X : \sup_{u \in T(\omega,z)} \mathrm{Re}\langle u, x - z\rangle + f(\omega, x, z) > 0\})$.*

Then

(a) *for each fixed $\omega \in \Omega$, the set*

$$\{x \in X : \sup_{y \in F(\omega,x)} [\inf_{u \in T(\omega,x)} \mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\}$$

is open in X;

(b) Graph $F \in \Sigma \otimes \mathcal{B}(X \times X)$*;*

(c) *there exists a measurable map $\phi : \Omega \to X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and*

$$\inf_{u \in T(\omega,\phi(\omega))} \mathrm{Re}\langle u, \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y) \le 0$$

for all $y \in F(\omega, \phi(\omega))$ and $\omega \in \Omega$.

Proof. (a) Fix $\omega \in \Omega$. Since X is a compact subset of the locally convex Hausdorff topological vector space E, and E^* is equipped with the strong topology, the function $\psi_1 : E^* \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $\psi_1(u, x, y) = \mathrm{Re}\langle u, x - y\rangle$ for each $(u, x, y) \in E^* \times X \times X$ is continuous. Since $T(\omega, \cdot) : X \to 2^{E^*}$ is upper semicontinuous with non-empty strongly compact values, by Theorem 1 of Aubin [4, p. 67], the function $\psi_2 : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $\psi_2(x, y) = \inf_{u \in T(\omega,x)} \mathrm{Re}\langle u, x - y\rangle$ is also lower semicontinuous. Thus $(x, y) \mapsto \inf_{u \in T(\omega,x)} \mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)$ is lower semicontinuous by (iii). As $F(\omega, \cdot) : X \to 2^X$ is lower semicontinuous with non-empty values, by Proposition III-19 in [5, p. 118], the map $x \mapsto \sup_{y \in F(\omega,x)} \inf_{u \in T(\omega,x)} [\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)]$ is also lower semicontinuous from X to $\mathbb{R} \cup \{-\infty, +\infty\}$ for each fixed $\omega \in \Omega$, so that the set

$$\Sigma(\omega) = \{x \in X : \sup_{y \in F(\omega,x)} \inf_{u \in T(\omega,x)} [\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\}$$

is open in X.

(b) Since F is random continuous with closed values, by Theorem 3.5 in [26, p. 57], we have $\operatorname{Graph} F \in \Sigma \otimes \mathcal{B}(X \times X)$.

(c) Since all hypotheses of Theorem 5.3 are satisfied, the conclusion follows.□

If both correspondences T and F are measurable, we have the following:

Theorem 5.5. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X be a non-empty bounded convex Polish subset of a locally convex Hausdorff topological vector space E. Suppose that*

(i) *$F : \Omega \times X \to 2^X$ is measurable such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is compact and continuous with non-empty compact convex values;*

(ii) *$T : \Omega \times X \to 2^{E^*}$ is measurable such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is compact and upper semicontinuous with non-empty strongly compact convex values;*

(iii) *$f : \Omega \times X \times X \to \mathbb{R}$ is measurable such that* (a) *for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X;* (b) *for each fixed $(\omega, x) \in \Omega \times X$, $f(\omega, x, x) = 0$ and $y \mapsto f(\omega, x, y)$ is lower semicontinuous and concave;*

(iv) *there exists a non-empty Polish subset E_0^* of E^* such that $T(\Omega \times X) \subset E_0^*$; and*

(v) *for each $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \setminus K(\omega)$ there exists $y \in \operatorname{co}(X_0(\omega) \cup \{x\})$ with $y \in \operatorname{co}(F(\omega, x) \cap \{z \in X : \sup_{u \in T(\omega, z)} \operatorname{Re}\langle u, x - z\rangle + f(\omega, x, z) > 0\})$.*

Then there exist measurable maps $\phi : \Omega \to X$ and $\rho : \Omega \to E^*$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$, $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \{\operatorname{Re}\langle \rho(\omega), \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)\} \le 0$$

for all $\omega \in \Omega$.

Proof. By Theorems 5.3 and 5.4, it remains to prove that

$$\{(\omega, x) \in \Omega \times X : \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} [\operatorname{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathcal{B}(X).$$

Since T and F are measurable, by Theorem 4.2(e) of Wagner [57], there exist two countable families of measurable maps $p_n : \Omega \times X \to X$ and $q_n : \Omega \times X \to E^*$ such that $F(\omega, x) = \operatorname{cl}\{p_n(\omega, x) : n = 1, 2, \dots\}$ and $T(\omega, x) = \operatorname{cl}\{q_n(\omega, x) : n = 1, 2, \dots\}$ for each $(\omega, x) \in \Omega \times X$.

We define a mapping $g_0 : E^* \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by $g_0(u, x, y) = \operatorname{Re}\langle u, x - y\rangle$ for each $(u, x, y) \in E^* \times X \times X$. Then g_0 is continuous so that g_1 is measurable. Therefore the function $g_0' : \Omega \times E^* \times X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $g_0'(\omega, u, x, y) = \operatorname{Re}\langle u, x - y\rangle + f(\omega, x, y)$ is also measurable for each

$(\omega, u, x, y) \in \Omega \times E^* \times X \times X$ since f is measurable. Fix $n \in \mathbb{N}$, note that $p_n : \Omega \times X \to X$ is measurable and f is measurable. Thus for each $j \in \mathbb{N}$, the function $g_j^n : \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $g_j^n(\omega, x) = \mathrm{Re}\langle q_j(\omega, x), x - p_n(\omega, x)\rangle + f(\omega, x, p_n(\omega, x))$ is also measurable by Lemma 3 in [39, p. 55]. Therefore the mapping $g_n : \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $g_n(\omega, x) = \inf_{j \in \mathbb{N}} g_j^n(\omega, x) = \inf_{j \in \mathbb{N}}\{\mathrm{Re}\langle q_j(\omega, x), x - p_n(\omega, x)\rangle + f(\omega, x, p_n(\omega, x))\}$ is measurable. Note that the mapping $g : \Omega \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $g(\omega, x) = \sup_{n \in \mathbb{N}} g_n(\omega, x)$ for each $(\omega, x) \in \Omega \times X$ is also measurable. Since for each $(\omega, x) \in \Omega \times X$, the mapping $y \mapsto f(\omega, x, y)$ is lower semicontinuous, we have

$$\begin{aligned}
&\{(\omega, x) \in \Omega \times X : \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} [\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\} \\
&= \{(\omega, x) \in \Omega \times X : \sup_{n \in \mathbb{N}} \inf_{j \in \mathbb{N}} [\mathrm{Re}\langle q_j(\omega, x), x - p_n(\omega, x)\rangle + f(\omega, x, p_n(\omega, x))] > 0\} \\
&= \{(\omega, x) : g(\omega, x) > 0\} \in \Sigma \otimes \mathcal{B}(X).
\end{aligned}$$

Therefore

$$\{(\omega, x) \in \Omega \times X : \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} [\mathrm{Re}\langle u, x - y\rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathcal{B}(X).$$

□

Let X in Theorem 5.5 be compact. Then we have:

Corollary 5.6. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X be a non-empty compact convex subset of a Banach space E whose dual space E^* is separable. Suppose that*

(i) *$F : \Omega \times X \to 2^X$ is measurable such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact convex values;*

(ii) *$T : \Omega \times X \to 2^{E^*}$ is measurable such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact and convex values;*

(iii) *$f : \Omega \times X \times X \to \mathbb{R}$ is measurable such that* (a) *for each fixed $(\omega, y) \in \Omega \times X, x \mapsto f(\omega, x, y)$ is lower semicontinuous on X;* (b) *for each fixed $(\omega, x) \in \Omega \times X, f(\omega, x, x) = 0$ and $y \mapsto f(\omega, x, y)$ is lower semicontinuous and concave.*

Then there exist measurable maps $\phi : \Omega \to X$ and $\rho : \Omega \to E^*$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$, $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \{\mathrm{Re}\langle \rho(\omega), \phi(\omega) - y\rangle + f(\omega, \phi(\omega), y)\} \le 0$$

for all $\omega \in \Omega$.

Let $f = 0$ in Corollary 5.6. Then we have the following:

COROLLARY 5.7. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X be a non-empty compact convex subset of a Banach space E whose dual space E^* is separable. Suppose that*

(i) *$F : \Omega \times X \to 2^X$ is measurable such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact convex values;*

(ii) *$T : \Omega \times X \to 2^{E^*}$ is measurable such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact convex values.*

Then there exist measurable maps $\phi : \Omega \to X$ and $\rho : \Omega \to E^*$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$, $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \mathrm{Re}\langle \rho(\omega), \phi(\omega) - y \rangle \le 0$$

for all $\omega \in \Omega$.

Theorem 5.4 is also a non-compact stochastic version of Theorem 4 of Shih and Tan in [44, p. 341] (and its improvements due to Kim [31, Theorem] and to Shih and Tan [45, Theorem 2, p. 69–70] (with $M = 0$)).

Theorem 5.4 generalizes Theorem of Tan [50, p. 326] in the following ways: (1) the set X need not be compact; (2) the correspondence T is upper semicontinuous instead of being continuous and (3) the function f need not be random continuous. In the case $F(x) = X$ and $T(x) = 0$ for each $x \in X$, Theorem 5.4 also improves Theorem 9.2.3 of Zhang [59, p. 304] with weaker continuity and measurability conditions. We also remark that our arguments used in proving the existence of solutions for generalized random quasi-variational inequalities in this section are different from those used by Tan [50] and Zhang [59], etc.

Quasi-variational inequalities and generalized quasi-variational inequalities have many applications in mathematical economics, game theory and optimization and other applied sciences (see e.g. [4], [5] and [6]). For sure, random quasi-variational inequalities and generalized random quasi-variational inequalities will also have many applications in random mathematical economics, random game theory and related fields.

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