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COMPACTNESS OF THE GREEN OPERATOR OF NONLINEAR DIFFUSION EQUATIONS: APPLICATION TO BOUSSINESQ TYPE SYSTEMS IN FLUID DYNAMICS

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Dedicated to Professor Jean Leray

1. Introduction

It is well-known that compactness arguments are very useful in the study of the existence and regularity of solutions of perturbed nonlinear evolution equations. In the first part of the paper we fix our attention on nonlinear diffusion problems of the type

$$(P_\beta) \qquad \begin{cases} v_t - \Delta\beta(v) = g & \text{in } Q_T \doteq (0,T)\times\Omega, \\ \beta(v) = 0 & \text{on } \Sigma_T \doteq (0,T)\times\partial\Omega, \\ v(0,x) = v_0(x) & \text{on } \Omega, \end{cases}$$

where Ω is an open regular bounded domain of $\mathbb{R}^N$ and β is a continuous non-decreasing function such that $\beta(0)=0$. Problem (P_β) arises in the study of heat conduction when the thermal conductivity depends on the temperature (see e.g. G. Díaz and J. I. Díaz [5] and its references). It also appears in many other physical contexts (see the general expositions by Aronson [2], Kalashnikov [8] and Vázquez [18]). In Section 2 we prove the compactness of the *Green type operators* $g \mapsto v$ and $g \mapsto \beta(v)$ for $v_0 \in L^1(\Omega)$ fixed, improving previous results by the authors (see Díaz and Vrabie [6], [7]).

As an application of the mentioned results we consider the Boussinesq type system

$$(S_{\beta F})\qquad \begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\pi = \mathbf{F}(v) & \text{in } Q_T,\\ v_t + (\mathbf{u}\cdot\nabla)v - \Delta\beta(v) = 0 & \text{in } Q_T,\\ \operatorname{div}\mathbf{u} = 0 & \text{in } Q_T,\\ \mathbf{u} = 0 \text{ and } \beta(v) = 0 & \text{on } \Sigma_T,\\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \text{ and } v(0,x) = v_0 & \text{in } \Omega, \end{cases}$$

where now we assume $N = 2$. The special case $\beta(s) = s$ and $\mathbf{F}(v) = (0, v - \widetilde{v})$, for some $\widetilde{v} \in \mathbb{R}$ fixed, arises in thermohydraulics (see, e.g., Temam [17, p. 129]). In these circumstances, the unknowns $\mathbf{u}, v$ and π represent the nondimensionalized velocity, temperature and pressure, respectively, of a fluid occupying Ω. The general case analyzed here could be interpreted as a mathematical model describing the movement of a fluid inside Ω, in which a diffusion process takes place simultaneously. According to the type of diffusion considered, $\mathbf{u}$ and π are again the nondimensionalized velocity and pressure of the fluid respectively, but v may represent either the concentration of a certain substance diffusing into the fluid, or its temperature. We remark that our results can be easily applied to more sophisticated systems arising, for instance, in the study of premixed flame models for the combustion of multicomponent mixtures of gases (see Manley, Marion and Temam [14]). We also remark that the most important particular case, of relevance in applications, corresponds to the choice of β either as a piecewise linear function (see Rulla [15] and Rodrigues [16]) or $\beta(s) = |s|^{m-1}s$ for some $m > 0$ (see references in the mentioned surveys). Our existence result for system $(S_{\beta F})$ concerns the cases of piecewise linear or *fast diffusion* operators, i.e. $\beta(s) = |s|^{m-1}s$ and $m \in (0,1]$. When $0 < m < 1$ the problem is of relevance in plasma physics (see references in G. Díaz and J. I. Díaz [5]). Finally, we point out that our existence result gives the additional regularity $(\mathbf{u}\cdot\nabla)v \in L^2(Q_T)$ which is of interest for nonlinear diffusion operators.

2. Compactness properties of the nonlinear diffusion equation

In this section we prove some compactness results for the nonlinear diffusion problem (P_β). Let Ω be a nonempty and bounded domain in $\mathbb{R}^N$ having the cone property and let $\beta : \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function with $\beta(0) = 0$. We define $B : \mathbb{R} \to \mathbb{R}$ by

$$B(r) = \int_0^r \beta(\tau)\,d\tau, \qquad \text{for each } r \in \mathbb{R}.$$

In what follows we say that a function $f : \mathbb{R}^k \to \mathbb{R}^l$ has a *sublinear growth* if there exist $a > 0$ and $b > 0$ such that

$$\|f(z)\| \leq a\|z\| + b, \qquad \text{for each } z \in \mathbb{R}^k.$$

We point out that if β has a sublinear growth, taking into account that β is nondecreasing and $\beta(0) = 0$, we easily deduce that B is nonnegative on $\mathbb{R}_+$ and the corresponding superposition operator associated with B maps bounded subsets in $L^2(\Omega)$ into bounded subsets in $L^1(\Omega)$.

It is well-known (see the mentioned surveys) that for each $g \in L^1(Q_T)$ and each $v_0 \in L^1(\Omega)$, there exists a unique function $v \in C([0,T]; L^1(\Omega))$ with $\beta(v) \in L^1(0;T; W_0^{1,1}(\Omega))$ satisfying (P_β) in the sense of distributions over Q_T. We shall refer to such a function v as to the *unique solution* of (P_β) corresponding to g and v_0, and we shall denote it by

$$v = \varphi(g, v_0). \tag{1}$$

The next compactness result has been proved in Díaz and Vrabie [6].

LEMMA 1. *If β is strictly increasing, then for each fixed $v_0 \in L^1(\Omega)$ and each weakly relatively compact subset G in $L^1(Q_T)$, the set*

$$\varphi(G, v_0) = \{\varphi(g, v_0);\ g \in G\}$$

is strongly relatively compact in $C([0,T]; L^1(\Omega))$.

The next consequence of Lemma 1 improves [7, Corollary 3.1].

COROLLARY 1. *If β is strictly increasing, then, for each fixed $v_0 \in L^2(\Omega)$ and each $p \in [1,2)$, the mapping $\varphi(\cdot, v_0) : L^2(Q_T) \to C([0,T]; L^1(\Omega))$ carries $L^2(Q_T)$ into $C([0,T]; L^p(\Omega))$ and is weakly-strongly sequentially continuous from the former to the latter. In addition, if $v_0 \in L^\infty(\Omega)$ then the result holds for any $p \in [1, \infty)$.*

PROOF. We start by recalling that if β is continuous nondecreasing and $\beta(0) = 0$ then, for each $p \in [1, \infty)$, each $v_0 \in L^p(\Omega)$ and each $g \in L^1(0,T; L^p(\Omega))$, $v = \varphi(g, v_0)$ satisfies

$$\|v(t)\|_{L^p(\Omega)} \leq \|v_0\|_{L^p(\Omega)} + \int_0^t \|g(\tau)\|_{L^p(\Omega)}\, d\tau, \tag{2}$$

for each $t \in [0,T]$ (see Bénilan [4]). Therefore, if $v_0 \in L^2(\Omega)$, $g \in L^2(Q_T)$ and $p \in [1,2)$, we have

$$v \in L^\infty(0,T; L^2(\Omega)) \subset L^\infty(0,T; L^p(\Omega)).$$

Now, let us assume by contradiction that v does not belong to $\mathcal{C}([0,T];L^p(\Omega))$. Then there exist $t \in [0,T]$ and $(t_n)_{n\in\mathbb{N}}$ in $[0,T]$ with $t_n \to t$ and $\varepsilon > 0$ such that

$$\varepsilon \leq \|v(t) - v(t_n)\|_{L^p(\Omega)}, \tag{3}$$

for each $n \in \mathbb{N}$. Since $v \in \mathcal{C}([0,T];L^1(\Omega))$, at least on a subsequence, we have

$$\lim_{n\to\infty} v(t_n) = v(t) \qquad \text{a.e. in } \Omega.$$

Inasmuch as $(v(t_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ and $p \in [1,2)$, from Ladyzhenskaya et al. [9, Lemma 2.2, p. 72], we conclude that on this subsequence $v(t_n) \to v(t)$ in $L^p(\Omega)$. But this contradicts (3). Thus $v \in \mathcal{C}([0,T];L^p(\Omega))$. In a very similar way, using Lemma 1, we may prove that for each fixed $v_0 \in L^2(\Omega)$ and each bounded subset G in $L^2(Q_T)$, $\varphi(G, v_0)$ is strongly relatively compact in $\mathcal{C}([0,T];L^p(\Omega))$.

Now, let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(Q_T)$ weakly convergent in this space to g. Since $\{\varphi(g_n, v_0); n \in \mathbb{N}\}$ is strongly relatively compact in $\mathcal{C}([0,T];L^p(\Omega))$, to complete the proof, it suffices to show that the set of all limit points of $(\varphi(g_n, v_0))_{n\in\mathbb{N}}$ in $\mathcal{C}([0,T];L^p(\Omega))$ contains only $\varphi(g, v_0)$. But this follows from the fact that the unique limit point of $(\varphi(g_n, v_0))_{n\in\mathbb{N}}$ in the sense of distributions over Q_T is $\varphi(g, v_0)$, and this completes the proof. A similar argument can be applied for any $p \in [1,\infty)$ if $v_0 \in L^\infty(\Omega)$. □

In order to prove the next compactness result, a simple lemma is needed.

LEMMA 2. *If β is continuous nondecreasing and $\beta(0) = 0$, then, for each $v_0 \in L^1(\Omega)$ with $B(v_0) \in L^1(\Omega)$, the unique solution w of (P_β) corresponding to $g = 0$, i.e. $w = \varphi(0, v_0)$, satisfies*

$$B(w(t)) \in L^1(\Omega), \qquad \textit{for each } t \in [0,T], \tag{4}$$

and

$$\|\nabla\beta(w(t))\|^2_{L^2(\Omega)} \leq \frac{1}{t-s}\|B(w(s))\|_{L^1(\Omega)}, \tag{5}$$

for each $0 \leq s < t \leq T$.

PROOF. Multiplying both sides in $w_t - \Delta\beta(w) = 0$ by $\beta(w)$, integrating over Ω and over $[s,t] \subset [0,T]$, we get

$$\int_s^t \int_\Omega \frac{dB(w)}{d\tau}\, dx\, d\tau + \int_s^t \int_\Omega \|\nabla\beta(w)\|^2\, dx\, d\tau = 0,$$

and thus

$$\|B(w(t))\|_{L^1(\Omega)} + \int_s^t \|\nabla\beta(w(\tau))\|^2_{L^2(\Omega)}\, d\tau = \|B(w(s))\|_{L^1(\Omega)}. \tag{6}$$

Since $B(v_0) \in L^1(\Omega)$, from (6) we easily get (4). Next, multiplying both sides in $w_t - \Delta\beta(w) = 0$ by $\beta'(w)w_t$, integrating over Ω and over $[s,t] \subset [0,T]$, we obtain

$$\int_s^t \int_\Omega w_\tau^2 \beta'(w)\, dx\, d\tau + \frac{1}{2}\int_s^t \frac{d}{d\tau}\int_\Omega \|\nabla\beta(w)\|^2\, dx\, d\tau = 0,$$

and consequently

$$\|\nabla\beta(w(t))\|^2_{L^2(\Omega)} \le \|\nabla\beta(w(s))\|^2_{L^2(\Omega)},$$

for each $0 \le s < t \le T$. From this inequality and (6) we deduce (5) and this completes the proof. □

LEMMA 3. *If β is continuous nondecreasing, with sublinear growth and $\beta(0) = 0$, then, for each fixed $v_0 \in L^2(\Omega)$ and each bounded subset G in $L^2(Q_T)$, the set*

$$\beta(\varphi) = \{\beta(v^g); v^g \doteq \varphi(g, v_0),\ g \in G\}$$

is strongly relatively compact in $C([0,T]; L^p(\Omega))$, provided $p \in [1,2)$ and, in fact, for any $p \in [1,\infty)$ if $v_0 \in L^\infty(\Omega)$.

PROOF. We denote by $S(t) : L^1(\Omega) \to L^1(\Omega)$, $t \ge 0$, the semigroup of nonexpansive mappings generated by $\Delta\beta$ on $L^1(\Omega)$, i.e.

$$S(t)v_0 = \varphi(0, v_0)(t),$$

for each $v_0 \in L^1(\Omega)$ and $t \ge 0$. Let $v_0 \in L^2(\Omega)$ be fixed and let G be a bounded subset in $L^2(Q_T)$. For each $g \in G$, we set $v^g = \varphi(g, v_0)$. Then, in view of (2), we deduce that the set

$$\{S(\lambda)v^g(t-\lambda);\ g \in G,\ t \in [0,T],\ \lambda \ge 0,\ t - \lambda \ge 0\}$$

is bounded in $L^2(\Omega)$. Moreover, from Vrabie [19, Lemma 2.3.1, p. 65], we have

$$\|v^g(t) - S(\lambda)v^g(t-\lambda)\|_{L^1(\Omega)} \le \int_{t-\lambda}^t \|g(\tau)\|_{L^1(\Omega)}\, d\tau$$

for each $g \in G$, $t \in [0,T]$ and $\lambda \ge 0$ with $t - \lambda \ge 0$. Thus

$$\|v^g(t) - S(\lambda)v^g(t-\lambda)\|_{L^1(\Omega)} \le \sqrt{\lambda|\Omega|}|G|,$$

for each $g \in G, t \in [0,T]$ and $\lambda \geq 0$ with $t - \lambda \geq 0$, where $|\Omega|$ is the Lebesgue measure of Ω and $|G| = \sup\{\|g\|_{L^2(Q_T)}; g \in G\}$. Hence

$$\lim_{\lambda \to 0} \|v^g(t) - S(\lambda)v^g(t-\lambda)\|_{L^1(\Omega)} = 0$$

for each $t \in (0,T]$, uniformly for $g \in G$. Since β is continuous and has sublinear growth, $\{\beta(v^g(t));\ g \in G,\ t \in [0,T]\}$ and $\{\beta(S(\lambda)v^g(t-\lambda);\ g \in G,\ t \in [0,T],\ \lambda \geq 0,\ t-\lambda \geq 0\}$ are bounded in $L^2(\Omega)$. The last relation in conjunction with [9, Lemma 2.2, p. 72] shows that

$$\lim_{\lambda \to 0} \|\beta(v^g(t)) - \beta(S(\lambda)v^g(t-\lambda))\|_{L^p(\Omega)} = 0, \tag{7}$$

for each $t \in (0,T]$, uniformly for $g \in G$, provided $p \in [1,2)$. Next, by (5), we have

$$\|\nabla\beta(S(\lambda)v^g(t-\lambda))\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda}\|B(v^g(t-\lambda))\|_{L^1(\Omega)},$$

for $g \in G$, $t \in (0,T]$ and $\lambda > 0$ with $t - \lambda \geq 0$. Since $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, this inequality along with (2) shows that $\{\beta(S(\lambda)v^g(t-\lambda));\ g \in G,\ \lambda > 0,\ t-\lambda \geq 0\}$ is relatively compact in $L^2(\Omega)$ for each $t \in (0,T]$. From (7) we then conclude that, for each $t \in [0,T]$, $\{\beta(v^g(t));\ g \in G\}$ is relatively compact in $L^p(\Omega)$, for $p \in [1,2)$.

We will now show that $\{\beta(v^g);\ g \in G\}$ is equicontinuous from $[0,T]$ into $L^2(\Omega)$. To this end, let $t \in [0,T]$ and let $\theta \in C_0^\infty(\Omega)$. We have

$$\int_\Omega [v^g(t+h) - v^g(t)]\theta\, dx = \int_t^{t+h}\int_\Omega \beta(v^g(\tau))\Delta\theta\, dx\, d\tau + \int_t^{t+h}\int_\Omega g\theta\, dx\, d\tau,$$

for each $g \in G$ and $h \in \mathbb{R}$ with $t + h \in [0,T]$. Consequently,

$$\lim_{h \to 0} |\langle v^g(t+h) - v^g(t), \theta\rangle_{L^2(\Omega)}| = 0$$

uniformly for $g \in G$. Since $C_0^\infty(\Omega)$ is densely imbedded in $L^2(\Omega)$ and $\{v^g;\ g \in G\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ (see (2)), the last relation shows that $\{v^g; g \in G\}$ is weakly equicontinuous from $[0,T]$ into $L^2(\Omega)$.

Next, we prove that $\{\beta(v^g); g \in G\}$ is strongly equicontinuous from $[0,T]$ into $L^p(\Omega)$, for $p \in [1,2)$, or equivalently, for $p \in (1,2)$. To this end, let $p \in (1,2)$ and let $A : D(A) \subset L^p(\Omega)$ be the realization of β^{-1} in $L^p(\Omega)$, i.e.

$$Aw = \{z \in L^p(\Omega);\ z(x) \in \beta^{-1}(w(x)) \text{ a.e. for } x \in \Omega\}$$

for each $w \in D(A)$, where $D(A) = \{w \in L^p(\Omega);\ \exists z \in L^p(\Omega),\ z(x) \in \beta^{-1}(w(x))$ a.e. for $x \in \Omega\}$. Since β is nondecreasing and continuous, β^{-1} is maximal monotone and thus A is m-accretive. Inasmuch as $p \in (1,2)$, the dual of $L^p(\Omega)$ is uniformly convex and thus, in view of Barbu [3, Proposition 3.5, p. 75], A is demiclosed, i.e. its graph is strongly-weakly sequentially closed in $L^p(\Omega)$. Hence, if $w_n \to w$ and $z_n \rightharpoonup z$ in $L^p(\Omega)$, and $w_n \in D(A)$, $z_n \in Aw_n$ for each $n \in \mathbb{N}$, then $w \in D(A)$ and $z \in Aw$.

Now, let us assume by contradiction that $\{\beta(v^g);\ g \in G\}$ is not strongly equicontinuous in $L^p(\Omega)$ at some $t \in [0,T]$. Then there exist $\varepsilon > 0$, $(g_n)_{n\in\mathbb{N}}$ in G, and $h_n \to 0$ such that $t + h_n \in [0,T]$ for each $n \in \mathbb{N}$, and

$$\varepsilon \leq \|\beta(v^{g_n}(t+h_n)) - \beta(v^{g_n}(t))\|_{L^p(\Omega)},$$

for each $n \in \mathbb{N}$. Without loss of generality, we may assume that there exists $v \in L^2(\Omega)$ such that $v^{g_n}(t) \rightharpoonup v$ in $L^2(\Omega)$ and also in $L^p(\Omega)$. Since $\{v^g;\ g \in G\}$ is weakly equicontinuous from $[0,T]$ into $L^2(\Omega)$, we easily conclude that $v^{g_n}(t+h_n) \to v$ in $L^p(\Omega)$. Now, let us recall that $\{\beta(v^{g_n}(t));\ n \in \mathbb{N}\}$ is strongly relatively compact in $L^p(\Omega)$ for $p \in [1,2)$. Consequently, on a subsequence at least, we have

$$\beta(v^{g_n}(t)) \to w \qquad \text{in } L^p(\Omega).$$

But $\beta(v^{g_n}(t)) \in D(A)$ and $v^{g_n}(t) \in A\beta(v^{g_n}(t))$, for each $n \in \mathbb{N}$. Inasmuch as A is demiclosed, we have $w \in D(A)$ and $v \in Aw$, i.e. $w = \beta(v)$. Similarly, one may show that, on a subsequence at least,

$$\beta(v^{g_n}(t+h_n)) \to \widetilde{w} = \beta(v) \qquad \text{in } L^p(\Omega).$$

Thus $\beta(v^{g_n}(t+h_n)) - \beta(v^{g_n}(t)) \to 0$ in $L^p(\Omega)$, thereby contradicting (8). This contradiction can be eliminated only if $\{\beta(v^g);\ g \in G\}$ is equicontinuous from $[0,T]$ into $L^p(\Omega)$.

Finally, by the Arzelà-Ascoli Theorem, we conclude that $\{\beta(v^g); g \in G\}$ is strongly relatively compact in $C([0,T]; L^p(\Omega))$, for each $p \in [1,2)$, and this completes the proof. If $v_0 \in L^\infty(\Omega)$ the maximum principle shows $v(t,\cdot) \in L^\infty(\Omega)$ and an easy modification yields the result for any $p \in [1,\infty)$. □

COROLLARY 2. *If β is continuous nondecreasing with sublinear growth and $\beta(0) = 0$, then, for each fixed $v_0 \in L^2(\Omega)$ and $p \in [1,2)$, the operator $g \mapsto \beta(v^g)$ maps $L^2(Q_T)$ into $C([0,T]; L^p(\Omega))$, and is weakly-strongly sequentially continuous from the former into the latter. The conclusion holds for any $p \in [1,\infty)$ if $v_0 \in L^\infty(\Omega)$.*

PROOF. In view of Lemma 3, $g \mapsto \beta(v^g)$ maps $L^2(Q_T)$ into $C([0,T]; L^p(\Omega))$, provided $p \in [1,2)$. Therefore, we have merely to show that this operator is weakly-strongly sequentially continuous from $L^2(Q_T)$ into $\mathcal{C}([0,T];L^p(\Omega))$, for $p \in [1,2)$. Thus, let $g_n \rightharpoonup g$ in $L^2(Q_T)$. Since, by Lemma 3, $\{\beta(v^{g_n}); n \in \mathbb{N}\}$ is strongly relatively compact in $\mathcal{C}([0,T];L^p(\Omega))$, to complete the proof it suffices to show that the set of all limit points of $(\beta(v^{g_n}))_{n\in\mathbb{N}}$ in $\mathcal{C}([0,T];L^p(\Omega))$ contains only $\beta(v^g)$. To prove this, let us assume that, on a subsequence, we have

$$\beta(v^{g_n}) \longrightarrow w \qquad \text{in } \mathcal{C}([0,T];L^p(\Omega)).$$

As $L^2(\Omega)$ is compactly imbedded in $H^{-1}(\Omega)$ and, for each $t \in [0,T]$, $\{v^{g_n}(t);\ n \in \mathbb{N}\}$ is bounded in $L^2(\Omega)$ (see (2)), this set is strongly relatively compact in $H^{-1}(\Omega)$. According to [19, Theorem 2.3.1, p. 61], the set $\{v^{g_n};\ n \in \mathbb{N}\}$ is strongly relatively compact in $\mathcal{C}([0,T];H^{-1}(\Omega))$. At this point let us observe that $g_n \rightharpoonup g$ in $L^2(Q_T)$ implies $g_n \rightharpoonup g$ in $L^2(0,T;H^{-1}(\Omega))$. Following the very same arguments as in [19, Corollary 2.3.1, p. 67], we deduce that $v^{g_n} \longrightarrow v^g$ strongly in $\mathcal{C}([0,T];H^{-1}(\Omega))$. Now, for each $t \in [0,T]$, at least on a subsequence, we have $v^{g_n}(t) \rightharpoonup v^g(t)$ in $L^2(\Omega)$, and also in $L^p(\Omega)$. Recalling that $\beta(v^{g_n}(t)) \longrightarrow w(t)$ in $L^p(\Omega)$, and reasoning as in the last part of the proof of Lemma 3, we deduce that $w(t) = \beta(v^g(t))$, for $t \in [0,T)$. Thus, the only limit point of $(\beta(v^{g_n}))_{n\in\mathbb{N}}$ in $\mathcal{C}([0,T];L^p(\Omega))$, for $p \in [1,2)$, is $\beta(v^g)$ and this completes the proof. The result for $p \in [1,\infty)$ is proved in a similar way. □

REMARK 1. Under the hypotheses of Corollary 2, we may prove that for each $g \in L^2(Q_T)$ and $p \in [1,2)$, $v^g \in \mathcal{C}([0,T];L^p(\Omega))$, and also that the operator $g \mapsto v^g$ is weakly-weakly sequentially continuous from $L^2(Q_T)$ into $\mathcal{C}([0,T];L^p(\Omega))$.□

3. An existence result for a Boussinesq type system

We start by introducing the functional setting of the system $(S_{\beta,F})$. As usual (see Leray [11], Lions [13], Temam [17]) we rewrite the Navier-Stokes equation in the space of divergence free vector fields. We recall that, in this way, we obtain a new system with only two unknown functions $\mathbf{u}$ and v. Namely, let us consider the function spaces

$$\begin{aligned}
\mathcal{C}_\sigma^\infty(\Omega) &= \{\mathbf{u} \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^2);\ \operatorname{div}\mathbf{u} = 0\};\\
H_\sigma(\Omega) &= \text{ the completion of } \mathcal{C}_\sigma^\infty(\Omega) \text{ in the } L^2(\Omega;\mathbb{R}^2)\text{-norm}\\
W_\sigma^{1,p}(\Omega) &= W_0^{1,p}(\Omega;\mathbb{R}^2) \cap H_\sigma(\Omega);\\
H_\sigma^2(\Omega) &= H^2(\Omega;\mathbb{R}^2) \cap W_\sigma^{1,2}(\Omega).
\end{aligned}$$

Let $P_\sigma : L^2(\Omega;\mathbb{R}^2) \longrightarrow H_\sigma(\Omega)$ be the orthogonal projection and let us define the Stokes operator $-\Delta_\sigma : H_\sigma^2(\Omega) \longrightarrow H_\sigma(\Omega)$ by

$$\Delta_\sigma \mathbf{u} = P_\sigma \Delta \mathbf{u}$$

for each $\mathbf{u} \in H^2_\sigma(\Omega)$. Applying the projection P_σ to both sides of the first equation of $(S_{\beta,F})$, observing that $P_\sigma \nabla\pi = 0$ and that $\operatorname{div}\mathbf{u} = 0$ implies $\mathbf{u} = P_\sigma \mathbf{u}$, we see that $(S_{\beta,F})$ may be rewritten as

$$(9) \qquad \begin{cases} \mathbf{u}_t + P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta_\sigma \mathbf{u} = P_\sigma \mathbf{F}(v) & \text{in } Q_T, \\ v_t + (\mathbf{u}\cdot\nabla)v - \Delta\beta(v) = 0 & \text{in } Q_T, \\ \mathbf{u} = 0 \ \text{ and } \ \beta(v) = 0 & \text{on } \Sigma_T, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \ \text{ and } \ v(0,x) = v_0(x) & \text{in } \Omega. \end{cases}$$

We may now proceed to the statement of the main results for problem $(S_{\beta F})$.

THEOREM 1. *Let Ω be a nonempty and bounded domain in $\mathbb{R}^2$ having the cone property and $\nu > 0$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that β^{-1} is a locally Lipschitz function. Let $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ be a continuous function. Then, for each $T > 0, \mathbf{u}_0 \in W^{1,2}_\sigma(\Omega)$ and $v_0 \in L^\infty(\Omega)$, problem (9) has at least one solution $(\mathbf{u}, v)$ in the following sense:*

$$(10) \qquad \mathbf{u} \in C([0,T]; H_\sigma(\Omega)) \cap L^\infty(0,T; W^{1,2}_\sigma(\Omega)) \cap L^2(0,T; H^2_\sigma(\Omega));$$

$$(11) \qquad \mathbf{u}_t, P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u} \in L^2(0,T; H_\sigma(\Omega));$$

$$(12) \qquad v \in C([0,T]; L^p(\Omega)), \qquad \textit{for each } p \in [1,2);$$

$$(13) \qquad \beta(v) \in C([0,T]; L^p(\Omega)) \cap L^2(0,T; H^1_0(\Omega)) \cap L^\infty(Q_T), \qquad \textit{for each } p \in [1,\infty);$$

$$(14) \qquad (\mathbf{u}\cdot\nabla)v \in L^2(Q_T),$$

and $(\mathbf{u}, v)$ satisfies (9) in the sense of distributions over Q_T.

REMARK 2. We do not know if Theorem 1 can be extended to the case of $\Omega \subset \mathbb{R}^3$. The main difficulty is to get suitable a priori estimates in order to prove (14), which is crucial in order to conclude the local existence (by means of our approach). We also point out that Theorem 1 remains valid, at least for some $T > 0$, for a general initial datum $v_0 \in L^2(\Omega)$ if we assume additionally that β^{-1} is a globally Lipschitz function. □

REMARK 3. Using similar arguments to those in Díaz and Vrabie [7] we may extend Theorem 1 to the case in which $\mathbf{F}$ is an upper semicontinuous multifunction with nonempty and compact convex values. We may also allow $\mathbf{F}$ to depend on $\mathbf{u}$ and we may replace the second equation in $(S_{\beta,F})$ by

$$v_t + (\mathbf{u}\cdot\nabla)v - \Delta\beta(v) \in G(\mathbf{u}, v) \qquad \text{in } Q_T,$$

where $G : \mathbb{R}^2 \to 2^{\mathbb{R}}$ is upper semicontinuous with nonempty and compact convex values. □

To simplify the exposition we will assume $\nu = 1$ in (9). We note that all the arguments used in this specific case are essentially the same as those needed for arbitrary $\nu > 0$.

The idea of proof in Theorem 1 is of topological nature and rests heavily upon Corollary 1. It consists in showing that a suitably defined operator has at least one fixed point whose existence implies the existence of a local solution of (9). We describe briefly this idea. First, let us fix $T^* > 0, r > 0$ and $\rho > 0$ (which will be chosen very precisely later on) and let us define the set $K = B_\sigma(0,1) \times B(0,\rho)$, where $B_\sigma(0,r)$ and $B(0,\rho)$ are the closed balls with center 0 and radius r and ρ, in $L^2(0,T^*;H_\sigma(\Omega))$ and $L^2(Q_{T^*})$ respectively. Next, let us observe that, for each $(\mathbf{f}, g) \in K$, each of the two problems

$$(15) \qquad \begin{cases} \mathbf{u}_t - \Delta_\sigma \mathbf{u} = \mathbf{f} & \text{in } Q_{T^*}, \\ \mathbf{u} = 0 & \text{on } \Sigma_{T^*}, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } \Omega, \end{cases}$$

and (P_β) has a unique solution $\mathbf{u} \in \mathcal{C}([0,T^*];H_\sigma(\Omega))$ and $v \in \mathcal{C}([0,T^*];L^1(\Omega))$. Moreover, since $v_0 \in L^\infty(\Omega)$ we can assume, without loss of generality, that $\mathbf{F}$ and β have sublinear growth and β^{-1} is globally Lipschitz. Indeed, by the maximum principle any function v satisfying

$$(P_{\beta,\mathbf{u}}) \qquad \begin{cases} v_t + (\mathbf{u}\cdot\nabla)v - \Delta\beta(v) = 0 & \text{in } Q_{T^*}, \\ \beta(v) = 0 & \text{on } \Sigma_{T^*}, \\ v(0,x) = v_0(x) & \text{on } \Omega, \end{cases}$$

when $\operatorname{div} u = 0$ must satisfy

$$\|v\|_{L^\infty(Q_{T^*})} \leq \|v_0\|_{L^\infty(\Omega)}$$

(see e.g. Rulla [15]). Then we can replace β over the intervals

$$(-\infty, -\|v_0\|_{L^\infty(\Omega)} - 1) \cup (\|v_0\|_{L^\infty(\Omega)} + 1, \infty)$$

by any continuous extension $\widetilde{\beta}$ of β having sublinear growth and with β^{-1} globally Lipschitz. So the solutions of $(P_{\beta,\mathbf{u}})$ and $(P_{\widetilde{\beta},\mathbf{u}})$ must coincide. A similar argument holds for $\mathbf{F}$. Now, let us define

$$Q(\mathbf{f}, g) \doteq (-P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u} + P_\sigma\mathbf{F}(v), -(\mathbf{u}\cdot\nabla)v),$$

where $\mathbf{u}$ and v correspond to $(\mathbf{f}, g)$ by means of (15) and (P_β). Clearly, whenever $(\mathbf{f}, g) \in K$ is a fixed point of $Q, (\mathbf{u}, v)$ satisfies (15) and (P_β) is a solution of (9). Hence, to complete the proof, it suffices to show that Q has at least one fixed point. To this end we will prove that, for some suitably chosen $T^* > 0$, $r > 0$ and $\rho > 0$, Q maps K into itself and is weakly-weakly sequentially continuous from K into K (both the domain and range being endowed with the weak topology of $L^2(0, T^*; H_\sigma(\Omega)) \times L^2(Q_{T^*})$). Since K is convex and weakly compact in $L^2(0, T^*; H_\sigma(\Omega)) \times L^2(Q_{T^*})$, by Arino, Gauthier and Penot's Fixed Point Theorem [1], Q has at least one fixed point in K.

Before showing how to choose $T^* > 0$, $r > 0$ and $\rho > 0$, some preliminaries are needed. First, let us define $\varphi : H_\sigma(\Omega) \to [0, \infty]$ by

$$\varphi(\mathbf{u}) = \begin{cases} \frac{1}{2}\sum_{i,j=1}^{2} \int_\Omega |\partial \mathbf{u}_i/\partial x_j|^2 dx & \text{if } \mathbf{u} \in W_\sigma^{1,2}(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

It is well-known that φ is convex, l.s.c. and proper and its subdifferential coincides with the Stokes operator, i.e.

$$\partial\varphi(\mathbf{u}) = -\Delta_\sigma \mathbf{u}$$

for each $\mathbf{u} \in D(\partial\varphi)$, where $D(\partial\varphi) = H_\sigma^2(\Omega)$. The next lemmas will be useful later.

LEMMA 4. *If $\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$ has sublinear growth there exist $k \in (0,1)$, $h > 0$, $c > 0$ and $d > 0$ such that, for each $\mathbf{u} \in H_\sigma^2(\Omega)$ and $v \in L^2(\Omega)$, we have*

$$(16) \quad \|-P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u} + P_\sigma \mathbf{F}(\mathbf{v})\|^2_{H_\sigma(\Omega)} \le k\|\Delta_\sigma \mathbf{u}\|^2_{H_\sigma(\Omega)} + h[\varphi(\mathbf{u})]^3 + c\|v\|^2_{L^2(\Omega)} + d.$$

PROOF. Since $\Omega \subset \mathbb{R}^2$ there exists $C > 0$, such that

$$\|P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u}\|^2_{H_\sigma(\Omega)} \le C\varphi(\mathbf{u})\|\mathbf{u}\|_{H_\sigma(\Omega)}\|\Delta_\sigma \mathbf{u}\|_{H_\sigma(\Omega)}$$

for each $\mathbf{u} \in H_\sigma^2(\Omega)$ (see Temam [17]). Since $W_\sigma^{1,2}(\Omega)$ is continuously imbedded in $H_\sigma(\Omega)$, there exists $C_1 > 0$ such that

$$\|\mathbf{u}\|_{H_\sigma(\Omega)} \le C_1\|\mathbf{u}\|_{W_\sigma^{1,2}(\Omega)} = C_1\sqrt{2}(\varphi(\mathbf{u}))^{1/2}$$

for each $\mathbf{u} \in W_G^{1,2}(\Omega)$. From the last two inequalities we get

$$(17) \qquad \|P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u}\|^2_{H_\sigma(\Omega)} \le m[\varphi(\mathbf{u})]^{3/2}\|\Delta_\sigma \mathbf{u}\|_{H_\sigma(\Omega)},$$

for each $\mathbf{u} \in H^2_\sigma(\Omega)$, where $m = \sqrt{2}CC_1$. In view of (17), we then have

$$\| - P_\sigma(\mathbf{u} \cdot \nabla)\mathbf{u} + P_\sigma \mathbf{F}(v)\|^2_{H_\sigma(\Omega)} \leq 2m[\varphi(\mathbf{u})]^{3/2}\|\Delta_\sigma \mathbf{u}\|_{H_\sigma(\Omega)} + 2\|P_\sigma \mathbf{F}(v)\|^2_{H_\sigma(\Omega)}$$

for each $\mathbf{u} \in H^2_\sigma(\Omega)$. But $\mathbf{F}$ has sublinear growth and thus there exist $a > 0$ and $b > 0$ such that $\|\mathbf{F}(v)\| \leq a|v| + b$ for each $v \in \mathbb{R}$. From this remark, the preceding inequality and Cauchy's inequality with $\varepsilon > 0$, we get

$$\begin{aligned}\| - P_\sigma(\mathbf{u} \cdot \nabla)\mathbf{u} &+ P_\sigma \mathbf{F}(v)\|^2_{H_\sigma(\Omega)} \\ &\leq 2\left\{\frac{\varepsilon}{2}\|\Delta_\sigma \mathbf{u}\|^2_{H_\sigma(\Omega)} + \frac{m}{2\varepsilon}[\varphi(\mathbf{u})]^3\right\} + 2[a\|v\|_{L^2(\Omega)} + b]^2 \\ &\leq \varepsilon\|\Delta_\sigma \mathbf{u}\|^2_{H_\sigma(\Omega)} + \frac{m}{\varepsilon}[\varphi(\mathbf{u})]^3 + 4a^2\|v\|^2_{L^2(\Omega)} + 4b^2\end{aligned}$$

for each $\mathbf{u} \in H^2_\sigma(\Omega)$ and $v \in L^2(\Omega)$. This inequality clearly shows that (16) holds with $k = \varepsilon \in (0, 1)$, $h = m/\varepsilon$, $\alpha = 4a^2$ and $d = 4b^2$, and this completes the proof. □

LEMMA 5. *For each* $\mathbf{u}_0 \in W^{1,2}_\sigma(\Omega)$ *and each* $r > 0$, *there exist two nondecreasing functions* $\gamma, \theta : (0, \infty) \to (0, +\infty)$, *with*

$$\lim_{\tau \to 0} \gamma(\tau) = 0,$$

and such that, for each $\tau > 0$, *and each* $\mathbf{f} \in L^2(0, \tau; H_\sigma(\Omega))$ *with* $\|\mathbf{f}\|_{L^2(0,\tau;H_\sigma(\Omega))} \leq r$, *the unique solution* $\mathbf{u}$ *of* (15) *satisfies*

$$\|\mathbf{u}\|_{L^\infty(Q_\tau;\mathbb{R}^2)} \leq \gamma(\tau) + \theta(\tau). \tag{18}$$

PROOF. In view of Solonnikov's Theorem (see e.g. von Wahl [20, Theorem III.1.1, p. 67]), there exists a nondecreasing function $l : (0, \infty) \to (0, \infty)$ such that, for each $\mathbf{u}_0 \in W^{1,2}_\sigma(\Omega)$ and each $\mathbf{f} \in L^2(0, \tau; H_\sigma(\Omega))$, the unique solution $\mathbf{u}$ of (15) satisfies

$$\begin{aligned}\|\mathbf{u}_t\|_{L^2(Q_\tau;\mathbb{R}^2)} + \|\mathbf{u}\|_{L^2(0,\tau;H^2_\sigma(\Omega))} + \|\mathbf{u}\|_{L^\infty(0,\tau;W^{1,2}_\sigma(\Omega))} \\ \leq l(\tau)[\|\mathbf{f}\|_{L^2(0,\tau;H_\sigma(\Omega))} + \sqrt{\varphi(\mathbf{u}_0)}].\end{aligned} \tag{19}$$

Hence the components of $\mathbf{u}$ belong to the space $W^{2,1}_2(Q_\tau)$ as defined in Ladyzhenskaya et al. [9, p. 5]. Since $\Omega \subset \mathbb{R}^2$ has the cone property, also from [9, Lemma 3.3, p. 80], $[W^{2,1}_2(Q_\tau)]^2 \subset L^\infty(Q_\tau; \mathbb{R}^2)$ and there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\|\tilde{\mathbf{u}}\|_{L^\infty(Q_\tau;\mathbb{R}^2)} \leq C_1\|\tilde{\mathbf{u}}\|_{L^2(Q_\tau;\mathbb{R}^2)} + C_2[\|\tilde{\mathbf{u}}_t\|_{L^2(Q_\tau;\mathbb{R}^2)} + \|\tilde{\mathbf{u}}\|_{L^2(0,\tau;H^2_\sigma(\Omega))}] \tag{20}$$

for each $\widetilde{\mathbf{u}} \in [W_2^{2,1}(Q_\tau)]^2$. From (19) and (20), we conclude that there exists $C_3 > 0$ such that

$$(21)\quad \|\mathbf{u}\|_{L^\infty(Q_\tau;\mathbb{R}^2)} \leq C_1\|\mathbf{u}\|_{L^2(0,\tau;H_\sigma(\Omega))} + C_3 l(\tau)[\|\mathbf{f}\|_{L^2(0,\tau;H_\sigma(\Omega))} + \sqrt{\varphi(\mathbf{u}_0)}].$$

Since

$$\|\mathbf{u}(t)\|_{H_\sigma(\Omega)} \leq \|\mathbf{u}_0\|_{H_\sigma(\Omega)} + \int_0^t \|\mathbf{f}(s)\|_{H_\sigma(\Omega)}\, ds$$

for each $t \in [0,\tau]$, and $W_\sigma^{1,2}(\Omega)$ is continuously imbedded in $H_\sigma(\Omega)$, there exists $C_4 > 0$ such that

$$(22)\qquad \|\mathbf{u}\|_{L^2(0,\tau;H_\sigma(\Omega))} \leq C_4\sqrt{\tau\varphi(\mathbf{u}_0)} + \tau\|\mathbf{f}\|_{L^2(0,\tau;H_\sigma(\Omega))}$$

for each $\mathbf{u}_0 \in W_\sigma^{1,2}(\Omega)$ and $\mathbf{f} \in L^2(0,\tau;H_\sigma(\Omega))$.

Now, let $\mathbf{u}_0 \in W_\sigma^{1,2}(\Omega)$ be fixed and let $r > 0$. We define $\gamma : (0,\infty) \to (0,\infty)$ by $\gamma(\tau) = C_1[C_4\sqrt{\tau\varphi(\mathbf{u}_0)} + \tau r]$, and $\theta : (0,\infty) \to (0,\infty)$ by $\theta(\tau) = C_3 l(\tau)[r + \sqrt{\varphi(\mathbf{u}_0)}]$. Clearly $\lim_{\tau\to 0}\gamma(\tau) = 0$, and by (21) and (22) we get (18), thereby completing the proof. □

LEMMA 6. *For each $v_0 \in L^2(\Omega)$ and each $\rho > 0$, there exists a nondecreasing function $\eta : (0,\infty) \to (0,\infty)$ with*

$$\lim_{\tau\to 0}\eta(\tau) = 0,$$

and such that, for each $g \in L^2(Q_\tau)$ with $\|g\|_{L^2(Q_\tau)} \leq \rho$, the unique solution v of (P_β) satisfies

$$(23)\qquad \|\nabla\beta(v)\|^2_{L^2(Q_\tau)} \leq \|B(v_0)\|_{L^1(\Omega)} + \eta(\tau).$$

PROOF. Multiplying both sides in (P_β) by $\beta(v)$, integrating over Ω and over $[0,\tau]$, we get

$$(24)\quad \|B(v(\tau))\|_{L^1(\Omega)} + \|\nabla\beta(v)\|^2_{L^2(Q_\tau)} = \|B(v_0)\|_{L^1(\Omega)} + \langle g, \beta(v)\rangle_{L^2(Q_\tau)}.$$

In view of (2), we easily deduce

$$(25)\qquad \|v(t)\|_{L^2(\Omega)} \leq \|v_0\|_{L^2(\Omega)} + \sqrt{\tau}\|g\|_{L^2(Q_\tau)},$$

for each $t \in [0,\tau]$. Recalling that β has sublinear growth, i.e. that there exist $\widetilde{a} > 0$ and $\widetilde{b} > 0$ such that

$$(26)\qquad |\beta(v)| \leq \widetilde{a}|v| + \widetilde{b},$$

for each $v \in \mathbb{R}$, from (24) and (25), we get

$$\|\nabla\beta(v)\|^2_{L^2(Q_\tau)} \le \|B(v_0)\|_{L^1(\Omega)} + \sqrt{\tau}\|g\|_{L^2(Q_\tau)}[\widetilde{a}\|v_0\|_{L^2(\Omega)} + \sqrt{\tau}\widetilde{a}\|g\|_{L^2(Q_\tau)} + \widetilde{b}\sqrt{|\Omega|}],$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. Hence (23) is satisfied by $\eta : (0,\infty) \to (0,\infty)$ defined as

$$\eta(\tau) = \sqrt{\tau}\rho[\widetilde{a}\|v_0\|_{L^2(\Omega)} + \sqrt{\tau}\widetilde{a}\rho + \widetilde{b}\sqrt{|\Omega|}],$$

for each $\tau > 0$, and this completes the proof. □

We may now proceed to the definition of the set K. To this end, let $\mathbf{u}_0 \in W^{1,2}_\sigma(\Omega)$ and $v_0 \in L^\infty(\Omega)$ and let us define $r > 0$ by

$$r^2 = 2\frac{1+3k}{1-k}[\varphi(\mathbf{u}_0) + 1], \tag{27}$$

where $k \in (0,1)$ is given by Lemma 4. Next, let us define $\rho > 0$ by

$$\rho^2 = M[1+\theta(1)]^2[\|B(v_0)\|_{L^1(\Omega)} + 1], \tag{28}$$

where M is the global Lipschitz constant of β^{-1} and $\theta : (0,\infty) \to (0,\infty)$ is given by Lemma 5, for $\mathbf{u}_0$ and $r > 0$ fixed as above. Finally, choose $T^* \in (0,1]$ satisfying

$$T^*\left(h\left[\varphi(\mathbf{u}_0) + \frac{r^2}{2}\right]^3 + \alpha(\|v_0\|_{L^2(\Omega)} + \sqrt{T^*}\rho)^2 + d\right) \le \frac{1-k}{2}r^2, \tag{29}$$

where $k \in (0,1)$, $h > 0$, $C > 0$ and $d > 0$ are given by Lemma 5, $r > 0$ is defined by (27) while ρ is defined by (28), and

$$\gamma(T^*) \le 1, \qquad \eta(T^*) \le 1. \tag{30}$$

Here $\gamma, \eta : (0,\infty) \to (0,\infty)$ are given by Lemmas 5 and 6 for $\mathbf{u}_0, v_0, r > 0$ and $\rho > 0$ fixed as above.

LEMMA 7. *Let $r > 0$, $\rho > 0$ and $T^* \in (0,1)$ satisfy (27)–(30), and let $K = B_\sigma(0,r) \times B(0,\rho)$ where $B_\sigma(0,1)$ and $B(0,\rho)$ are the closed balls with center 0 and radius r and ρ, in $L^2(0,T^*;H_\sigma(\Omega))$ and $L^2(Q_{T^*})$ respectively. Then, for each $(\mathbf{f}, g) \in K$, the unique solution $(\mathbf{u}, v)$ of (15) and (P_β) satisfies*

$$(-P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u} + P_\sigma\mathbf{F}(v), -(\mathbf{u}\cdot\nabla)v) \in K.$$

PROOF. First, let us recall that (15) may be rewritten as

$$\begin{cases} \frac{d\mathbf{u}}{dt}(t) + \partial\varphi(\mathbf{u}(t)) = \mathbf{f}(t), & 0 \le t \le T^*, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases}$$

Multiplying both sides by $\partial\varphi(\mathbf{u}(t))$ we get

$$\frac{d}{dt}\varphi(\mathbf{u}(t)) + \|\partial\varphi(\mathbf{u}(t))\|^2_{H_\sigma(\Omega)} \le \|\partial\varphi(\mathbf{u}(t))\|_{H_\sigma(\Omega)}\|\mathbf{f}(t)\|_{H_\sigma(\Omega)}$$

a.e. for $t \in (0, T^*)$, and

$$\varphi(\mathbf{u}(t)) + \frac{1}{2}\int_0^t \|\partial\varphi(\mathbf{u}(\tau))\|^2_{H_\sigma(\Omega)}\, d\tau \le \|\varphi(\mathbf{u}_0)\| + \frac{1}{2}\int_0^t \|\mathbf{f}(\tau)\|_{H_\sigma(\Omega)}\, d\tau,$$

for each $t \in [0, T^*]$. From this inequality we obtain both

$$\int_0^{T^*} \|\Delta_\sigma \mathbf{u}\|^2_{H_\sigma(\Omega)}\, d\tau \le 2\varphi(\mathbf{u}_0) + r^2 \tag{31}$$

and

$$\varphi(\mathbf{u}(t)) \le \varphi(\mathbf{u}_0) + \frac{r^2}{2}, \tag{32}$$

for each $t \in [0, T^*]$. From (31) we deduce that, for each $\varepsilon \in (0, 1/2)$, we have

$$(1 - 2\varepsilon)\|\Delta_\sigma \mathbf{u}\|^2_{L^2(0,T^*;H_\sigma(\Omega))} \le 2\varphi(\mathbf{u}_0) + r^2.$$

Taking $\varepsilon = (1-k)/(2(1+3k)) \in (0, 1/2)$, after some standard calculations involving (27), we get

$$k\|\Delta_\sigma \mathbf{u}\|^2_{L^2(0,T^*;H_\sigma(\Omega))} \le \frac{1+k}{2}r^2. \tag{33}$$

From Lemma 4, (32), (33) and (25), we deduce

$$\begin{aligned} &\|-P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u} + P_\sigma\mathbf{F}(v)\|^2_{L^2(0,T^*;H_\sigma(\Omega))} \\ &\qquad \le k\|\Delta_\sigma \mathbf{u}\|^2_{L^2(0,T^*;H_\sigma(\Omega))} + h\int_0^{T^*} [\varphi(\mathbf{u}(t))]^3\, dt + C\|v\|^2_{L^2(Q_{T^*})} + T^* d \\ &\qquad \le \frac{1+k}{2}r^2 + T^*\left\{h\left[\varphi(\mathbf{u}_0) + \frac{r^2}{2}\right]^3 + C(\|v_0\|_{L^2(\Omega)} + \sqrt{T^*}\rho)^2 + d\right\}. \end{aligned}$$

In view of (29), this inequality shows that

$$\|-P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u} + P_\sigma\mathbf{F}(v)\|^2_{L^2(0,T^*;H_\sigma(\Omega))} \le r^2. \tag{34}$$

But (34) shows that $-P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u}+P_\sigma\mathbf{F}(v)\in B_\sigma(0,r)$ for each $(\mathbf{f},g)\in K$. Next, let us observe that if M is the global Lipschitz constant of β^{-1} then

$$\|(\mathbf{u}\cdot\nabla)v\|^2_{L^2(Q_{T^*})}\leq M\|\mathbf{u}\|^2_{L^\infty(Q_{T^*};\mathbb{R}^2)}\|\nabla\beta(v)\|^2_{L^2(Q_{T^*})}$$

and thus, by (18) and (23), we get

$$\|(\mathbf{u}\cdot\nabla)v\|^2_{L^2(Q_{T^*})}\leq M[\gamma(T^*)+\theta(T^*)]^2[\|B(v_0)\|_{L^1(\Omega)}+\eta(T^*)].$$

Taking into account that for $T^*\in(0,1]$, $\theta(T^*)\leq\theta(1)$, and using (30) and (28), we conclude that

$$\|(\mathbf{u}\cdot\nabla)v\|^2_{L^2(Q_{T^*})}\leq M[1+\theta(1)]^2[\|B(v_0)\|_{L^1(\Omega)}+1]=\rho^2.$$

Thus $(\mathbf{u}\cdot\nabla)v\in B(0,\rho)$ and, along with (34), this completes the proof. □

PROOF OF THEOREM 1. In view of Lemma 7 we may define the operator $Q:K\to K$ by

$$Q(\mathbf{f},g)\doteq(-P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u}+P_\sigma\mathbf{F}(v),-(\mathbf{u}\cdot\nabla)v),\qquad(\mathbf{f},g)\in K,$$

where $\mathbf{u}$ and v satisfy (15) and (P_β) respectively. In order to apply Arino, Gauthier and Penot's Fixed Point Theorem [1], we only have to show that Q is weakly-weakly sequentially continuous. To this end, let $((\mathbf{f}_n,g_n))_{n\in\mathbb{N}}$ be a sequence in K such that

$$\mathbf{f}_n\rightharpoonup\mathbf{f}\quad\text{in } L^2(0,T^*;H_\sigma(\Omega)),\quad g_n\rightharpoonup g\quad\text{in } L^2(Q_{T^*}).$$

In view of Vrabie [19, Corollary 2.3.2, p. 68], we have

$$\mathbf{u}_n\to\mathbf{u}\qquad\text{in } C([0,T^*];H_\sigma(\Omega)),$$

where $\mathbf{u}_n$ is the solution of (15) corresponding to $\mathbf{u}_0$ and $\mathbf{f}_n$, while $\mathbf{u}$ is the solution of (15) corresponding to $\mathbf{u}_0$ and $\mathbf{f}$. We also have

$$P_\sigma(\mathbf{u}_n\cdot\nabla)\mathbf{u}_n\rightharpoonup P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u}\qquad\text{in } L^2(0,T^*;H_\sigma(\Omega)),$$

and, by Corollary 1, for each $p\in[1,\infty)$, $v_n\to v$ in $C([0,T^*];L^p(\Omega))$. Here v_n is the solution of (P_β) corresponding to v_0 and g_n, while v is the solution of (P_β) corresponding to v_0 and g.

Next, we will show that

$$(\mathbf{u}_n\cdot\nabla)v_n\rightharpoonup(\mathbf{u}\cdot\nabla)v\qquad\text{in } L^2(Q_{T^*}).\tag{35}$$

Assuming the contrary, on a subsequence at least, we have $(\mathbf{u}_n \cdot \nabla)v_n \to w$ in $L^2(Q_{T^*})$ where $w \neq (\mathbf{u} \cdot \nabla)v$. At this point let us observe that we may assume without loss of generality that $\mathbf{u}_n \to \mathbf{u}$ a.e. in Q_{T^*} and $\nabla v_n \to \mathbf{z}$ in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$. Consequently, $w = \mathbf{u} \cdot \mathbf{z}$. But $\beta(v_n) \to \beta(v)$ in $C([0,T^*]; L^p(\Omega))$ and β strictly increasing imply that $\mathbf{z} = \nabla v$. But this contradicts the initial assumption, and hence (35) holds.

Now, we will prove that

$$\mathbf{F}(v_n) \to \mathbf{F}(v) \qquad \text{in } L^2(Q_{T^*}) \times L^2(Q_{T^*}). \tag{36}$$

To this end, let us assume by contradiction that this is not the case. Then, at least on a subsequence, we must have

$$\mathbf{F}_0(\beta(v_n)) \to \widetilde{\mathbf{F}} \qquad \text{in } L^2(Q_{T^*}) \times L^2(Q_{T^*}),$$

where $\widetilde{\mathbf{F}} \neq \mathbf{F}_0(\beta(v))$. From Corollary 2 we may assume (taking a subsequence if necessary) that $\mathbf{F}(v_n) \to \mathbf{F}(v)$ a.e. in Q_{T^*}. But in view of [9, Lemma 2.3, p. 72], it follows that $\widetilde{\mathbf{F}} = \mathbf{F}(v)$. This contradiction can be eliminated only if (36) holds.

Since P_σ is linear continuous, from (36) we easily deduce that

$$P_\sigma \mathbf{F}(v_n) \rightharpoonup \mathbf{P}_\sigma \mathbf{F}(v) \qquad \text{in } L^2(0,T^*; H_\sigma(\Omega)).$$

Thus Q is weakly-weakly sequentially continuous from K into K, and since K is weakly compact in $L^2(0,T^*; H_\sigma(\Omega)) \times L^2(Q_{T^*})$, by virtue of Arino, Gauthier and Penot's Fixed Point Theorem [1], Q has at least one fixed point $(\mathbf{f}, g) \in K$, and this completes the proof of the existence of a local solution. From the boundedness of v we can assume, without loss of generality, that $\mathbf{F}$ is bounded on $\mathbb{R}$. Then, by a known result on the Navier-Stokes equation in the space dimension two, each noncontinuable solution of (9) must be defined on $[0,\infty)$. Since (10)–(14) follow from Lemma 4, (19) and Lemmas 3 and 7, the proof of Theorem 1 is complete. □

REFERENCES

[1] O. ARINO, S. GAUTHIER AND J. P. PENOT, *A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations*, Funkcial. Ekvac. **27** (1984), 273–279.

[2] D. G. ARONSON, *The porous mediun equation*, Nonlinear diffusion problems, Lecture Notes in Math. (A. Fasano and M. Primicerio, eds.), vol. 1224, Springer-Verlag, 1986, pp. 12–46.

[3] V. BARBU, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, 1976.

[4] PH. BÉNILAN, *Equations d'évolution dans un espace de Banach quelconque et applications*, Thèse, Orsay, 1972.

[5] G. DÍAZ AND J. I. DÍAZ, *Finite extinction time for a class of non-linear parabolic equations*, Comm. Partial Differential Equations **4** (1979), 1213–1231.

[6] J. I. DÍAZ AND I. I. VRABIE, *Propriétés de compacité de l'opérateur de Green généralisé pour l'équation des milieux poreux*, C. R. Acad. Sci. Paris **309** (1989), 221–223.

[7] ———, *Existence for reaction diffusion systems: A compactness method approach*, J. Math. Anal. Appl. (1994).

[8] A. S. KALASHNIKOV, *Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations*, Russian Math. Surveys **42** (1987), 76–87.

[9] O. A. LADYZHENSKAYA, V. A. SOLONNIKOV AND N. N. URAL'TSEVA, *Linear and Quasi-linear Equations of Parabolic Type*, Transl. Math. Monographs vol. **23**, Amer. Math. Soc., 1968.

[10] J. LERAY, *Etude de diverses équations intégrales non linéaires et de quelques problèmes que posent l'hydrodynamique*, J. Math. Pures Appl. **12** (1933), 1–82.

[11] ———, *Essai sur le mouvement plan d'un liquide visqueux que limitent des parois*, J. Math. Pures Appl. **13** (1934), 331–418.

[12] ———, *La théorie des points fixes et ses applications en Analyse*, Proc. Int. Congress Math. 1950, vol. 2, Amer. Math. Soc., 1952, pp. 202–208.

[13] J.-L. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, 1969.

[14] O. MANLEY, M. MARION AND R. TEMAM, *Equations of combustion in the presence of complex chemistry*, Indiana Univ. Math. J. **42** (1993), 941–967.

[15] J. RULLA, *Weak solutions to Stefan problems with prescribed convection*, SIAM J. Math. Anal. **18** (1987), 1784–1800.

[16] J. F. RODRIGUES, *Weak solutions for thermoconvective flows of Boussinesq-Stefan type*, Pitman Res. Notes in Math., vol. 279, Longman, 1992, pp. 93–116.

[17] R. TEMAM, *Navier-Stokes Equations*, North-Holland, 1977.

[18] J. L. VÁZQUEZ, *An introduction to the mathematical theory of the porous medium equation*, Shape Optimization and Free Boundaries (M. C. Delfour, ed.), Kluwer, 1992, pp. 347–389.

[19] I. I. VRABIE, *Compactness Methods for Nonlinear Evolutions*, Longman Scientific & Technical, 1987.

[20] W. VON WAHL, *The Equations of Navier-Stokes and Abstract Parabolic Equations*, Vieweg, 1985.

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