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INFINITE-DIMENSIONAL SINGULARITY THEORY AND BIFURCATION FROM EQUILIBRIA FOR CERTAIN NONLINEAR EVOLUTION EQUATIONS

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Dedicated to Jean Leray

1. Introduction

In this paper we look at parabolic partial differential equations from a new point of view. We regard a parabolic system as arising from an elliptic system by adding time dependence. We inquire if it is possible to use the detailed analysis of the resulting elliptic system to study the time-dependent situation. Moreover, we focus on special classes of solutions of the time-dependent problem, namely, periodic solutions.

A new feature of our viewpoint is an attempt to use the new infinite-dimensional theory of singularities of maps for nonlinear dynamical systems defined by nonlinear evolution equations. In fact, we regard a nonlinear evolution equation subject to appropriate boundary conditions as a mapping between two infinite-dimensional spaces. By focusing attention on periodic time-dependence we are able to assume the mapping in question is a Fredholm operator of index zero. Once this fact has been established, we are able to use the infinite-dimensional theory of singularities to determine a very explicit type of bifurcation at a singular point.

In this paper we carry out the detailed analysis of the above idea which was proposed in a paper of Berger and Schechter [5]. We consider the bifurcation of

T-periodic solutions from equilibria for the nonlinear evolution problem

$$\begin{aligned} u_t - Lu - f(x,u) &= g(x,t) \qquad \text{in } \Omega \times \mathbb{R}, \\ u &= 0 \qquad \text{on } \partial\Omega \times \mathbb{R}. \end{aligned} \tag{1}$$

with a given T-periodic smooth forcing term $g(x,t)$. The corresponding elliptic partial differential equation is

$$\begin{aligned} -Lu - f(x,u) &= g(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

The content of this paper is as follows: We consider equation (1) for Ω being an arbitrary bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and L a uniformly elliptic formally self-adjoint second order differential operator defined on Ω. The function $f(x,u)$ is specialized to be a C^2 real-valued convex function with f_u bounded and

$$0 < \lim_{s \to -\infty} \frac{f(x,s)}{s} < \lambda_1, \qquad \lambda_1 < \lim_{s \to +\infty} \frac{f(x,s)}{s} < \lambda_2, \tag{3}$$

where λ_1, λ_2 are the lowest two eigenvalues of $-L$ relative to Ω.

More explicitly, we define our real Hilbert spaces X and Y of T-periodic functions in t to be $X = W_{1,2}[(0,T), H]$ with the norm

$$\|u\|_X = \left\{ \int_{\Omega \times (0,T)} [u^2 + |u_t|^2 + |u_x|^2] \right\}^{1/2},$$

and $Y = L_2[(0,T), H]$ with the norm

$$\|u\|_Y = \left\{ \int_{\Omega \times (0,T)} [u^2 + |u_x|^2] \right\}^{1/2}.$$

Here $H = W^0_{1,2}(\Omega)$. Also we define a nonlinear operator $B : X \to Y$ by

$$\langle B(u), \phi \rangle_Y = \int_{\Omega \times (0,T)} [u_t \phi - uL\phi - f(x,u)\phi]$$

and a nonlinear operator $A : H \to H$ by

$$\langle A(u), \phi \rangle_H = \int_\Omega [uL\phi + f(x,u)\phi].$$

We prove the following theorems.

THEOREM 1. *The operator B regarded as a C^1 mapping between the real Hilbert spaces X and Y is a nonlinear Fredholm operator of index zero.*

THEOREM 2. *Regular points of the mapping A are regular points of the mapping B. Moreover, singular points of A are singular points of B and, for any $u \in H$,*

$$\dim \ker A'(u) = \dim \ker B'(u). \tag{4}$$

THEOREM 3. *Any singular point u of A is an infinite-dimensional Whitney fold for B, provided B is regarded as a mapping between X and Y.*

THEOREM 4. *For $g(x,t)$ smooth, restricted to a small neighborhood of a singular value of A in Y, the nonlinear evolution equation has exactly 2, 1 or 0 smooth, real T-periodic solutions in an appropriate neighborhood of the associated singular point of A.*

We look for a weak solution of equation (1) and try to use the duality method which enables us to conduct the detailed analysis of the bifurcations of periodic solutions.

In Section 2, we will give a short account of singularity theory. In Section 3, by choosing appropriate function spaces we will prove that the induced nonlinear operator is Fredholm of index zero. In Section 4, we will show the local properties of the induced operators.

2. Background from singularity theory

Let us begin by recalling some simple facts of singularity theory. For more details we refer to Berger [2] and Berger and Church [3]. Let X and Y be Banach subspaces of a Hilbert space Z and $F : X \to Y$ be a C^1 operator. By $F' : X \to L(X,Y)$ we denote the Fréchet derivative of the operator F. A linear operator $L : X \to Y$ is called *Fredholm* if (a) the range of L is closed in Y, and (b) the subspaces $\ker L$ and coker L are finite-dimensional. The *index* of L is

$$\text{index } L = \dim \ker L - \dim \text{coker } L.$$

We say a C^1 operator $F : X \to Y$ is *Fredholm* if $F'(u) \in L(X,Y)$ is a Fredholm operator for any $u \in X$. Actually, the index of $F'(u)$ does not depend on u, so that we may speak of the index of F.

Let $F : X \to Y$ be a C^1 operator. We say that $u \in X$ is a *singular point* of F if $\ker F'(u)$ contains a nontrivial element. Otherwise u is said to be a *regular point* of F. The set of singular points will be denoted by $S(A)$. We investigate what happens when mappings have singularities. The first two types of singularities are fold and cusp singularities (see Whitney [6]). Let us introduce the concept of infinite-dimensional Whitney fold.

DEFINITION 5. Let $u \in X$ be a singular point of F. We call u an *infinite-dimensional fold point* of F if the following conditions hold:

(i) F is Fredholm with index 0;

(ii) $\dim \ker F'(u) = 1$;

(iii) for some (and hence for any) nonzero element $e \in \ker F'(u)$,

$$\langle F''(u)(e,e), e\rangle_Z \neq 0.$$

The following proposition gives a nice description of the behavior of a mapping in a neighborhood of an infinite-dimensional fold point. This description is invariant under local coordinate change.

THEOREM 6 (BERGER AND CHURCH [3]). *Let u be an infinite-dimensional fold point of F. Then there exists a Banach space E such that F is locally C^∞ equivalent at u to the map $\phi : \mathbb{R} \times E \to \mathbb{R} \times E$ given by*

$$\phi(t,v) = (t^2, v), \qquad t \in \mathbb{R},\ v \in E.$$

3. Function spaces and properties of nonlinear operators

Let $H = W^0_{1,2}(\Omega)$. Denote by $W_{1,2}[(0,T),H]$ the Hilbert space of all real functions u on $\Omega \times \mathbb{R}$ that are T-periodic in t with the norm

$$\|u\|_1 = \left\{\int_Q \left[u^2 + |u_t|^2 + |u_x|^2\right]\right\}^{1/2}$$

for $Q = \Omega \times (0,T)$. By $L_2[(0,T),H]$ we mean the Hilbert space of all real functions u on $\Omega \times \mathbb{R}$ that are T-periodic in t with the norm

$$\|u\|_0 = \left\{\int_Q \left[u^2 + |u_x|^2\right]\right\}^{1/2}.$$

We define a solution of (1) as follows:

DEFINITION 7. By a *solution* of problem (1) we mean a function $u \in W_{1,2}[(0,T),H]$ which satisfies

$$\int_Q [u_t\phi - uL\phi - f(x,u)\phi] = \int_Q g\phi \tag{5}$$

for any $\phi \in L_2[(0,T),H]$.

We define $X = W_{1,2}[(0,T),H]$, $Y = L_2[(0,T),H]$.

Now for $u \in H \equiv W^0_{1,2}(\Omega)$ the formula

$$I_0(\psi) = \int_\Omega [uL\psi + f(x,u)\psi]$$

defines a bounded linear functional for $\psi \in H$ and the Riesz Representation Theorem yields $I_0(\psi) = \langle A(u),\psi\rangle_H$, giving an operator $A : H \to H$ satisfying

$$\langle A(u),\psi\rangle_H = \int_\Omega [uL\psi + f(x,u)\psi]. \tag{6}$$

For fixed $u \in W_{1,2}[(0,T),H]$ the formula

$$I(\phi) = \int_Q [u_t\phi - uL\phi - f(x,u)\phi]$$

defines a bounded linear functional for $\phi \in L_2[(0,T),H]$. This yields an operator $B : X \to Y$ such that

$$\langle B(u),\phi\rangle_Y = \int_Q [u_t\phi - uL\phi - f(x,u)\phi]. \tag{7}$$

It is clear that the operator B is well defined. Actually, we can write

$$B : X \to Y, \qquad B(u) = u_t - Lu - f(x,u),$$

with $D(B) = W_{1,2}[(0,T),H]$. If we denote the inner product and norm of H as

$$(u,v) = (u,v)_{W^0_{1,2}(\Omega)}, \qquad ||u||^2 = (u,u),$$

the inner product and norm in Y can also be written as

$$((u,v)) = \int_0^T (u,v)dt, \qquad |||u|||^2 = ((u,u)).$$

A linear operator related to B, called B_0, is defined by

$$B_0 u = u_t - Lu : X \to Y. \tag{8}$$

It is clear that the operators B and B_0 are well defined. Moreover, the operator B has the following property:

THEOREM 8. *The operator $B : X \to Y$ defined by (7) is a C^1 nonlinear Fredholm operator of index zero.*

PROOF. (a) In fact, we can define a linear operator $U : X \to Y$ by

$$((U(h), \phi)) = \int_Q [h_t\phi - hL\phi - f_u(x,u)h\phi].$$

It is clear that U is well defined. We compute

$$((B(u+h) - B(u) - Uh, \phi)) = -\int_Q [f(x, u+h) - f(x,u) - f_u(x,u)h]\phi.$$

Since f is smooth we know that $\|B(u+h) - B(u) - Uh\| = o(h)$. This means that the derivative of B at a point u is well defined and is given by

$$B'(u)v = v_t - Lv - f_u(x,u)v.$$

We note that the asymptotic properties (3) imply that $f_u(x,t) \to 0$ as $|t| \to \infty$, so that $f_u(x,t)$ is uniformly bounded on $(-\infty, +\infty)$. Using this fact we now show that as $u_n \to u$ in X, $B'(u_n) \to B'(u)$ in the space of bounded linear operators on X.

In fact,

$$|f_u(x,u_n) - f_u(x,u)| \leq (\sup f_{uu}(x,s))|u_n - u|,$$

so that if $u_n \to u$ in X, then $f_u(x,u_n) \to f_u(x,u)$ in measure over Q, and for fixed $v \in X$ of norm 1 in X,

$$\|B'(u_n)v - B'(u)v\| \leq \|f_u(x,u_n)v - f_u(x,u)v\|_{L_2(Q)} \to 0 \tag{9}$$

by the Lebesgue dominated convergence theorem. This last inequality shows that

$$B'(u_n)v \to B'(u)v,$$

and it remains to show that the convergence is uniform in v. We prove this by contradiction. Suppose the convergence is not uniform in v. Then there is a sequence $\{v_n\}$ of norm 1 functions in X and an absolute constant $\delta > 0$ such that

$$\|(f_u(x,u_n) - f_u(x,u))v_n\|_{L_2(Q)} \geq \delta > 0.$$

Using Rellich's lemma we may assume that $\{v_n\}$ is strongly convergent in Y. Thus

$$\begin{aligned}\|(f_u(x,u_n) - f_u(x,u))v_n\|_{L_2(Q)} \leq\ & \|f_u(x,u_n)(v_n - v)\|_{L_2(Q)} \\ &+ \|(f_u(x,u_n) - f_u(x,u))v\|_{L_2(Q)} \\ &+ \|f_u(x,u)(v_n - v)\|_{L_2(Q)}.\end{aligned}$$

Consequently, by (9),

$$\|(f_u(x,u_n)-f_u(x,u))v_n\|_{L_2(Q)} \to 0,$$

contrary to $\delta > 0$.

(b) We now show that the operator B so defined is a Fredholm operator of index zero. Using $B'(u)v = B_0 v - f_u(x,u)v$ we first show that B_0 is invertible. Indeed, we have

$$B_0^{-1}w(t) = e^{tL}(1-e^{TL})^{-1}Kw(T) + Kw(t)$$

where

$$Kw(t) = \int_0^t e^{-(s-t)L}w(s)\,ds.$$

In fact, since $D(B_0) = D(B)$ and from the inequality

$$(-Lu,u) \geq \alpha\|u\|^2, \qquad u \in D(L),\ \alpha > 0,$$

and $B'(u)v = v_t - Lv - f_u(x,u)v = B_0 v - f_u(x,u)v$ we find that

$$((B_0 u, u)) \geq \alpha|||u|||^2, \qquad u \in D(B),$$

by integrating $(u_t - Lu, u)$ over the interval $[0,T]$.

If we write

$$y(t) = Kw(t) = \int_0^t e^{-(s-t)L}w(s)\,ds$$

then

(10) $$y'(t) = e^{tL}w(0) + Kw'(t) \quad \text{and} \quad \|y(t)\| \leq t^{1/2}|||w|||.$$

Now defining an operator M by

$$Mw(t) = e^{tL}(1-e^{TL})^{-1}Kw(T) + Kw(t),$$

we find that $B_0 M w = w$. In fact, it is clear that

$$Mw(0) = (1-e^{TL})^{-1}Kw(T),$$
$$Mw(T) = e^{TL}(1-e^{TL})^{-1}Kw(T) + Kw(T) = Mw(0).$$

We rewrite $y(t)$ as

$$\begin{aligned} y(t) &= \int_0^t e^{-(s-t)L}\left[w(0) + \int_0^s w'(r)\,dr\right] ds \\ &= w(0)\int_0^t e^{-(s-t)L}\,ds + \int_0^t \left[\int_r^t e^{-(s-t)L}\,ds\right] w'(r)\,dr \end{aligned}$$

to obtain

$$-Ly(t) = (1 - e^{TL})w(0) + \int_0^t [1 - e^{-(r-t)L}]w'(r)\,dr$$
$$= w(t) - e^{TL}w(0) - Kw'(t) = w(t) - y'(t).$$

We actually have the estimate

$$\|Mw(t)\| \leq (1 + \|1 - e^{TL}\|)T^{1/2}|||w|||.$$

Using the fact that L^{-1} is compact, the Ascoli-Arzelá Theorem and the estimate

$$\|y(t) - y(s)\| \leq C_\theta |t - s|^\theta |||w|||, \qquad 0 < \theta < 1/2,$$

we find that K is a compact operator from Y to $C[(0,T), W_{1,2}(\Omega)]$. It follows that M is a compact operator from Y to X and $Mf_u(x,u)$ is also a compact operator from Y to X. Now for any $u \in D(B)$, $B'(u)$ is a Fredholm operator of index zero since $B'(u)$ can be written as

$$B'(u)v = B_0 v - f_u(x,u)v = B_0(v - Mf_u(x,u)v).$$

Thus B is a nonlinear Fredholm operator of index zero. □

4. Local behavior of nonlinear operators

To use the detailed analysis of fold singularities of elliptic operators to study the bifurcations of T-periodic solutions of evolution equations we need to know the relationship between T-periodic solutions of evolution equations and solutions of elliptic equations. The connection is the following:

THEOREM 9. *If $g(x,t) = g(x)$ in* (1), *then the T-periodic solutions of* (1) *are solutions of* (2).

PROOF. Set $Q = \Omega \times [0,T]$. We multiply both sides of (1) with u_t and then integrate over Q to get

$$\int_Q [u_t^2 - u_t Lu - f(x,u)u_t] = \int_Q g(x)u_t. \tag{11}$$

Since u is a T-periodic function, we have

$$\int_Q gu_t = \int_\Omega g(x) \int_0^T u_t = 0, \tag{12}$$

$$\int_Q f(x,u)u_t\,dt = \int_\Omega \int_0^T \frac{\partial}{\partial t}\left[\int_0^u f(x,s)\,ds\right] = 0, \tag{13}$$

and

$$\int_Q u_t Lu = \frac{1}{2}\int_\Omega \int_0^T \frac{\partial}{\partial t}\left(\frac{uLu}{2}\right) = 0. \tag{14}$$

Putting (12)–(14) into (11) gives $\int_Q u_t^2(x,t) = 0$. So $u(x,t) = u(x)$ and u is a solution of (2). □

The next theorem concerns the relationship between the singular points of the nonlinear operators A and B and the dimensions of the kernels of A' and B'.

THEOREM 10. *If $u \in H$ is a regular point of A, then u is a regular point of B. If $u \in H$ is a singular point of A, then u is a singular point of B. Moreover, for $u \in H \cap S(A)$,*

$$\dim\ker A'(u) = \dim\ker B'(u). \tag{15}$$

PROOF. (1) Let $u \in H$ be a regular point of A. Suppose that for some $v \in X$ such that $B'(u)v = 0$, we have

$$\int_Q [v_t\phi - vL\phi - f_u(x,u)v\phi] = 0 \tag{16}$$

for every $\phi \in L_2[(0,T), H]$. We want to show that $v \equiv 0$. Choosing $\phi = v_t$ in (16) we get

$$\int_Q [v_t^2 - v_t Lv - f_u(x,u)vv_t] = 0. \tag{17}$$

Since v is T-periodic in t and u is independent of t we have

$$\int_Q vLv_t = \frac{1}{2}\int_\Omega \int_0^T \frac{\partial}{\partial t}(vLv) = 0,$$

$$\int_Q f_u(x,u)vv_t = \int_\Omega f_u(x,u)\int_0^T \frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) = 0.$$

Putting these into (17) we get

$$\int_Q v_t^2 = 0,$$

which means that $v(x,t) = v(x)$ and v satisfies

$$\int_\Omega vL\left(\int_0^T \phi\right) + f_u(x,u)v\left(\int_0^T \phi\right) = 0.$$

But u is a regular point of A so $v \equiv 0$. Hence u is a regular point of B.

(2) Suppose $u \in H$ is a singular point of A. Then there exists a nontrivial element $v \in H$ such that

$$\int_\Omega [vL\psi + f_u(x,u)v\psi] = 0 \tag{18}$$

for all $\psi \in H$. Now $v \in H$ is independent of t, so that

$$\begin{aligned}((B'(u)v,\phi)) &= \int_Q [v_t\phi - vL\phi - f_u(x,u)v\phi] \\ &= -\int_\Omega \left[vL\left(\int_0^T \phi\right) + f_u(x,u)v\left(\int_0^T \phi\right)\right].\end{aligned}$$

Since $\psi = \int_0^T \phi \in H$, from (18) we get $((B'(u)v,\phi)) = 0$. So u is a singular point of B.

(3) We now start to prove (15). Using Theorem 8 we can write every $v \in \ker B'(u)$ as

$$v(x,t) = \sum_{k=1}^n c_k(t)v_k(x) + w(x,t) \tag{19}$$

where $c_1, \dots, c_n$ are weakly differentiable functions and $v_1, \dots, v_n$ is an orthonormal basis for $\ker A'(u)$, i.e.

$$\langle v_j, v_k\rangle_{L_2(\Omega)} = \delta_{jk}, \tag{20}$$

and w is orthogonal to $\ker A'(u)$ for all t in L_2 sense, i.e.

$$\langle w, v_j\rangle_{L_2(\Omega)} = 0 \qquad (j = 1, \dots, n) \tag{21}$$

for all t. To prove (15) it suffices to prove

$$c_k = \text{const.} \qquad (k = 1, \dots, n) \tag{22}$$

and

$$w(x,t) \equiv 0. \tag{23}$$

We first note that

$$c_k(t) = \int_\Omega v(x,t)v_k(x)\,dx \qquad \text{for } k = 1, \dots, n$$

and $c_k(t)$ is T-periodic in t. Since

$$v_t - Lv - f_u(x,u)v = 0,$$

we obtain

$$(v_t, v_k) = (v_t - A'(u)v, v_k) + (A'(u)v, v_k) = 0$$

by using the self-adjointness of $A'(u)$ and the fact that $v \in \ker B'(u)$. That is,

$$\frac{d}{dt}(v, v_k) = 0$$

and so

$$(v, v_k) = c_k = \text{const.} \qquad \text{for } k = 1, \dots, n.$$

Now since $v \in \ker B'(u)$ and v is independent of t we find that

$$A'(u)w = A'(u)v = -v_t = -w_t$$

and so

$$(A'(u)w, w) = -(w_t, w) = -\frac{1}{2}\frac{d}{dt}\|w\|^2.$$

But $w \perp \ker B'(u)$ for each fixed t, so we have

$$(A'(u)w, w) \geq c_0\|w\|^2, \qquad c_0 > 0.$$

Thus

$$c_0\|w\|^2 \leq -\frac{1}{2}\frac{d}{dt}\|w\|^2,$$

which implies $\|w\|^2 \equiv 0$ for any t. Hence $w(x,t) = 0$. □

LEMMA 11.

(a) *For $u \in H \cap S(A)$, condition* (iii) *in Definition* 5 *of a fold point reads*

$$\int_\Omega f_{uu}(x,u)e(u)^3 \neq 0, \tag{24}$$

where $e(u)$ is the solution of

$$\begin{aligned} Le + f_u(x,u)e &= 0 \qquad \text{in } \Omega, \\ e &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{25}$$

(b) *For $u \in X \cap S(B)$, condition* (iii) *in Definition* 5 *reads*

$$\int_0^T \int_\Omega f_{uu}(u)\tilde{e}(u)^3 \neq 0, \tag{26}$$

where $\widetilde{e}(u)$ is the T-periodic solution of

$$\begin{aligned} \widetilde{e}_t - L\widetilde{e} - f_u(x,u)\widetilde{e} &= 0 \qquad \text{in } \Omega \times \mathbb{R}, \\ \widetilde{e} &= 0 \qquad \text{in } \partial\Omega \times \mathbb{R}. \end{aligned} \tag{27}$$

Proof. We only give a proof for the case (b) since the proof for (a) is similar. From the definition of B we compute

$$B'(u)\widetilde{e} = \widetilde{e}_t - L\widetilde{e} - f_u(x,u)\widetilde{e} = 0,$$

since $\widetilde{e}$ is a T-periodic solution of (27). This means that $\widetilde{e} \in \ker B'(u)$. Now taking the Fréchet derivative of B' we get

$$\langle B''(u)(\widetilde{e},\widetilde{e}),\widetilde{e}\rangle_Y = -\int_Q f_{uu}(x,u)\widetilde{e}^3, \tag{28}$$

which is (26). □

We now have the following theorem regarding the type of the singular point of B.

Theorem 12. *Any $u \in H \cap S(A)$ is an infinite-dimensional Whitney fold for B, provided B is regarded as a mapping between X and Y.*

Proof. (a) We know that B is a nonlinear Fredholm operator of index zero and from Theorem 10,

$$\dim\ker B'(u) = \dim\ker A'(u) = 1.$$

Let $u \in H \cap S(A)$. From Berger and Church [3], [4] we know that u is a Whitney fold point of A. From the definition there exists a nonzero element $e \in \ker A'(u)$ such that

$$(A''(u)(e,e),e) \neq 0.$$

From Lemma 11(a), this can be written as

$$\int_\Omega f_{uu}(x,u)e^3 \neq 0. \tag{29}$$

Now since e is independent of t we know $e \in \ker B'(u)$. Integrating on both sides of (29) over $[0,T]$ gives

$$\int_0^T \int_\Omega f_{uu}(x,u)e^3 \neq 0, \tag{30}$$

and from Lemma 11(b) we know u is a Whitney fold point of B. □

For the multiplicity of T-periodic solutions of the nonlinear evolution equation (1) we have the following theorem.

THEOREM 13. *For $g(x,t)$ smooth, in a small neighborhood of a singular value of A in Y, the nonlinear evolution equation* (1) *has exactly* 2, 1 *or* 0 *smooth, real T-periodic solutions in X in an appropriate neighborhood of the associated singular point of A.*

PROOF. This follows from the regularity of weak solutions and the canonical normal form for an infinite-dimensional fold singularity. The regularity of weak solutions is obtained by using the standard bootstrap argument. □

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