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# ON PARABOLIC PROBLEMS GENERATED BY SOME SYMMETRIC FUNCTIONS OF THE EIGENVALUES OF THE HESSIAN

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*Dedicated to Jean Leray*

## 1. Introduction

This paper is a sequel to [14] and [15]. We study the first initial-boundary value problem in a bounded domain $Q_T = \Omega \times (0,T) \subset \mathbb{R}^{n+1}$ for the equations

$$M_m[u] \equiv -u_t + F_m(u_{xx}) = g, \qquad m = 2,\dots,n, \tag{1}$$

where $F_m(u_{xx}) = S_m(u_{xx})^{1/m}$ and $S_m(u_{xx})$ is the sum of all principal $m$-th order minors of the Hessian $u_{xx}$. Alternatively, $S_m$ can be defined as follows:

$$S_m(u_{xx}) = \sum_{i_1<\dots<i_m} \lambda_{i_1}(u)\dots\lambda_{i_m}(u),$$

where $\lambda_k(u)$, $k = 1,\dots,n$, are the eigenvalues of $u_{xx}$. For $m = 1$ equation (1) is the well-studied heat equation. Its stationary part $F_1(u_{xx}) = \Delta u$ is linear and totally uniformly elliptic, that is, uniformly elliptic on the whole space $C^2(\overline{\Omega})$. For $m > 1$ all these properties fail and one has to describe the domains of ellipticity of the equations

$$F_m(u_{xx}) = \Psi. \tag{2}$$

Equations (2) are obviously elliptic on the set of strongly convex (or concave) functions. In the case $m = n$ this set appeared to be the very set of solvability of the Dirichlet problem for (2) (see [1], [2], [4], [8]–[11], [21], [22]).

First, the study of equations (2) with $m = 2, \dots , n-1$, was also restricted to the set of convex functions. But the domain of ellipticity strictly extends when $m$ decreases from $n$ to 1. The reasonable choice of these domains leading to solvability of the Dirichlet problem was suggested in papers [12] and [13] for equations (2) and in [3] for some generalizations of (2).

Less is done for fully nonlinear parabolic equations. Most results concern totally parabolic equations (see [16], [25], [26]). The non-totally parabolic equations studied so far include the equations $u_t \det u_{xx} = f$ ([16], [24]) and the equations describing deformations of closed compact surfaces caused by their curvatures ([23], [5]).

To proceed with equations (1), following [12] let us introduce the subsets

$$K_m = \{A \in M_s^{n\times n} \,:\, S_k(A) \equiv \operatorname{spur}_k A > 0,\ k = 1, \dots , m\},$$

where $M_s^{n\times n}$ is the set of all symmetric $n \times n$ matrices. It was shown in [12] that $K_m$ is a connected set containing the unit matrix $I$. We shall keep the same notation $K_m$ for the set of functions $u$ from $C^2(\overline{\Omega})$, with Hessian $u_{xx}(x)$ belonging to $K_m$ for all $x \in \overline{\Omega}$. The set $K_m$ is the set of ellipticity for $K_m$ ([12]) and it appears to be the "natural" set of solvability of the Dirichlet problem for equations (2) with $\Psi > 0$ ([3], [12], [13]).

We seek solutions $u$ of (1) in $\overline{Q}_T$ which belong to $K_m$ for any $t \in [0, T]$ and satisfy the condition

$$u - \varphi = 0 \qquad \text{on } \partial' Q_T, \tag{3}$$

where $\partial' Q_T = \partial'' Q_T \cup \Omega(0)$, $\partial''\Omega_T = \{(x,t) \,:\, x \in \partial\Omega,\ t \in [0,T]\}$, and $\Omega(0) = \{(x,t) \,:\, x \in \overline{\Omega},\ t = 0\}$ We call such solutions *admissible*. In (3), $\varphi$ is an arbitrary sufficiently smooth function of $(x,t) \in \overline{Q}_T$ belonging to $K_m$ for $t = 0$ and satisfying the necessary conditions of compatibility with $g$ on the set $\Gamma_0 = \{(x,t) \,:\, x \in \partial\Omega,\ t = 0\}$.

In order to describe our requirements on $\partial\Omega$, we relate to every point $x^0 \in \partial\Omega$ cartesian coordinates "corresponding to $x^0$". This means that the axis $x_n$ is directed along the inner normal $\nu$ to $\partial\Omega$ at $x^0$. In some neighborhood of $x^0$ the surface $\partial\Omega$ is the graph of a function $\omega$, i.e.,

$$\partial\Omega = \{x = (\widetilde{x}, x_n) \,:\, \widetilde{x} = (x_1, \dots , x_{n-1}),\ |\widetilde{x}| < \varepsilon,\ x_n = \omega(\widetilde{x})\}.$$

For convenience we choose the tangent axes $x_1, \dots, x_{n-1}$ in such a way that

$$x_n = \omega(\widetilde{x}) = \frac{1}{2}\sum_{k=1}^{n-1} \omega_{kk}(x^0)x_k^2 + O(|\widetilde{x}|^3), \qquad |\widetilde{x}| < \varepsilon. \tag{$4_1$}$$

Our hypothesis on $\partial\Omega$ is

$$(\omega_{kl}(x^0) \equiv \omega_{kk}(x^0)\delta_l^k) \in K_{m-1} \subset M_s^{(n-1)\times(n-1)}. \tag{$4_2$}$$

Assume also that $\omega \in H^{4+\alpha}(\overline{B}_\varepsilon(x^0))$ for any $x^0 \in \partial\Omega$ and some $\varepsilon > 0$.

The main result of this paper is

THEOREM 1. *The problem* (1), (3) *has a unique admissible solution* $u$ *belonging to the Hölder space* $H^{4+\alpha,2+\alpha/2}(\overline{Q}_T)$ *if the following conditions are satisfied:*

(a) $g \in H^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$, $\varphi \in H^{4+\alpha,2+\alpha/2}(\overline{Q}_T)$, $\varphi|_{t=0} \in K_m$, *$g$ and $\varphi$ satisfy on $\Gamma_0$ the compatibility conditions up to the second order;*

(b) *the boundary $\partial\Omega$ is a surface of class $H^{4+\alpha}$ satisfying* ($4_2$);

(c) *$g$ and $\varphi$ satisfy*

$$\min\{\min_{x\in\Omega} F_m(\varphi_{xx}(x,0));\ \min_{Q_T} g + \min_{\partial' Q_T} u_t\} - \frac{a}{2}d^2 \equiv \nu_2 > 0, \tag{5}$$

*where $d$ is the radius of the smallest ball $B_d(x^0)$ containing $\Omega$,* $a = \max\{0; \nu_1^{-1} \max_{Q_T} g_t\}$ *and* $\nu_1 = (C_n^m)^{1/m}$.

*The condition* (c) *can be replaced by the following condition* (c′):

$$g_{xx}(x,t) \le 0, \qquad (F_m(\varphi_{xx}))_{xx}|_{t=0} \le 0, \qquad \min_{\partial' Q_T}(u_t + g) \equiv \widetilde{\nu}_2 > 0. \tag{6}$$

The uniqueness in the class of admissible solutions is proved in a standard way since $K_m$ is a convex set and (1) is parabolic on $K_m$. Namely, the difference $v = u' - u''$ of two admissible solutions to the problem (1), (3) is a classical solution to the linear parabolic equation

$$-v_t + a_{ij}v_{x_ix_j} = 0,$$

where

$$a_{ij}\xi_i\xi_j = \int_0^1 \frac{\partial F_m(u^\tau_{xx})}{\partial u^\tau_{ij}}\bigg|_{u^\tau = \tau u'' + (1-\tau)u'} d\tau\, \xi_i\xi_j > 0 \qquad \text{for } |\xi| = 1.$$

As soon as $v|_{\partial' Q_T} = 0$ the Hopf Theorem leads to $v \equiv 0$.

To prove the existence of a solution $u$ we include the problem (1), (3) in the family of problems

$$(7)\qquad M[u^\tau]=g^\tau,\quad u^\tau-\varphi^\tau|_{\partial''Q_T}=0,\quad u^\tau-\varphi^0|_{\Omega(0)}=0,\qquad \tau\in[0,1],$$

where $g^\tau=\tau g+(1-\tau)g^0$, $g^0\equiv F_m(\varphi^0_{xx})$, $\varphi^0=\varphi^0(x)\equiv\varphi(x,0)$ and $\varphi^\tau=\tau\varphi+(1-\tau)\varphi^0$.

The problem (7) for $\tau=0$ has the admissible solution $u^0(x,t)=\varphi^0(x)$ and for any $\tau\in[0,1]$ satisfies all conditions of Theorem 1, which can be easily verified. Therefore, an a priori estimate of the Hölder norm $\|u\|^{(4+\alpha,2+\alpha/2)}_{Q_T}$ of any admissible solution $u$ to problem (1), (3) under conditions (a)–(c) or (a), (b), (c′) will provide the solvability of the problem. But it is known that the norm of $u$ in $H^{4+\alpha,2+\alpha/2}(\overline{Q}_T)$ can be estimated by the norm of $u$ in $H^{2+\beta,1+\beta/2}(\overline{Q}_T)$ with some $\beta>0$ in view of some results on linear uniformly parabolic equations with Hölder coefficients (see, for example, [6], [17]). So our task is to get a majorant for $\|u\|^{(2+\beta,1+\beta/2)}_{Q_T}$ with some $\beta>0$. A pecularity of this problem is due to the fact that the quadratic form $j(u;\xi)=\frac{\partial F_m(u_{xx})}{\partial u_{ij}}\xi_i\xi_j$ with $|\xi|=1$ is positive on any compact subset of $K_m$ but has zero lower bound on all $K_m$. We shall prove that there are some positive minorant and majorant for $j(u;\xi)$ with any $|\xi|=1$ and admissible solution $u$. We start with a weaker information on $j(u;\xi)$:

$$(8)\qquad \sum_{i=1}^n\frac{\partial F_m(u_{xx})}{\partial u_{ii}}\ge\binom{n}{m}^{1/m}\equiv\nu_1.$$

This holds for any $u\in K_m$, and follows from the inequality

$$(9)\qquad \frac{\partial}{\partial a_{ij}}S_m(A)b_{ij}\ge mS_m(A)^{1-1/m}S_m(B)^{1/m},$$

valid for arbitrary matrices $A=(a_{ij})$ and $B=(b_{ij})$ in $K_m$ ([7]).

We introduce some abbreviations. For any $v\in C^2(\Omega)$ we write $v_i=v_{x_i}$, $v_\gamma=v_i\cos(\gamma,x_i)$, where $\gamma$ is any unit vector in $\mathbb{R}^n$, $v_{\gamma\gamma}=v_{ij}\cos(\gamma,x_i)\cos(\gamma,x_j)$.

Let us now fix the number $m$ in (1) and an admissible solution $u$ of problem (1), (3), and introduce the functions (of $(x,t)\in\overline{Q}_T$) $F=F_m(u_{xx})$, $F_{ij}=F_{ij}(u_{xx})=\frac{\partial}{\partial u_{ij}}F_m(u_{xx})$ and the linear operator $L_u=-\partial_t+F_{ij}\partial^2_{x_ix_j}$ corresponding to this solution $u=u(x,t)$, $(x,t)\in\overline{Q}_T$.

Differentiating (1) with respect to $t$ and $\gamma$ gives rise to the equations

$$(10_1)\qquad L_u(u_t)=g_t,$$

$$(10_2)\qquad L_u(u_\gamma)=g_\gamma,$$

$$(10_3)\qquad L_u(u_{\gamma\gamma})+\frac{\partial^2F}{\partial u_{ij}\partial u_{kl}}u_{\gamma ij}u_{\gamma kl}=g_{\gamma\gamma}.$$

## 2. Estimation of the $C^{2,1}(\overline{Q}_T)$ norm of $u$

By (8)

$$L_u\left(\frac{1}{2}(x-x^0)^2\right) = \sum_{i=1}^{n} F_{ii} \geq \nu_1 \tag{11}$$

for any $x^0 \in \mathbb{R}^n$. Therefore

$$L_u\left(u_t - \frac{a}{2}(x-x^0)^2\right) = g_t - a\sum_{i=1}^{n} F_{ii} \leq 0 \tag{12$_1$}$$

and

$$L_u\left(u_t + \frac{b}{2}(x-x^0)^2\right) = g_t + b\sum_{i=1}^{n} F_{ii} \geq 0 \tag{12$_2$}$$

with $a$ defined in Theorem 1 and $b = \max\{0; \nu_1^{-1} \max Q_T(-g_t)\}$. If $g_{xx}(x,t) \leq 0$ in $\overline{Q}_T$, then

$$L_u(u_t + g) = g_t + L_u(g) = F_{ij}g_{ij} \leq 0 \qquad \text{in } \overline{Q}_T. \tag{13}$$

These inequalities and the Hopf Theorem for the parabolic operator $L_u$ give

PROPOSITION 1. *The estimates*

$$\frac{a}{2}(x-x^0)^2 + \min_{\partial' Q_T} u_t - \frac{a}{2}d^2 \leq u_t(x,t) \leq -\frac{b}{2}(x-x^0)^2 + \max_{\partial' Q_T} u_t + \frac{b}{2}d^2 \tag{14}$$

*hold for any admissible solution $u$ to problem* (1), (3). *In* (14),

$$a = \max\{0; \nu_1^{-1} \max_{Q_T} g_t\}, \qquad b = \max\{0; \nu_1^{-1} \max_{Q_t}(-g_t)\}$$

*and $x^0$ is the center of a ball $B_d(x^0)$ of radius $d$ containing $\Omega$. If*

$$\min_{Q_T} g + \min_{\partial' Q_T} u_t - \frac{1}{2}ad^2 \equiv \nu_3 > 0, \tag{15}$$

*then*

$$u_t + g \geq \nu_3 > 0 \qquad \textit{in } \overline{Q}_T.$$

*If $g_{xx} \leq 0$ in $\overline{Q}_T$, then*

$$u_t + g \geq \min_{\partial' Q_T}(u_t + g), \tag{16}$$

*and if, additionally,*

$$\min_{\partial' Q_T}(u_t + g) \equiv \widetilde{\nu}_2 > 0, \tag{17}$$

*then*

$$u_t + g \geq \widetilde{\nu}_2 > 0 \qquad \textit{in } \overline{Q}_T. \tag{18}$$

The estimate for $\max_{Q_T}|u_t|$ implies an estimate for $\max_{Q_T}|u|$. Now we proceed with estimating $\max_{Q_T}|u_x|$.

Hopf's Theorem and (11) reduce the estimation of $\max_{Q_T}|u_x|$ to estimating $\max|u_x|$ on $\partial'' Q_T$. Indeed,

$$L_u\left[\frac{c_1}{2}(x-x^0)^2 \pm u_j\right] = c_1\sum_{i=1}^n F_{ii} \pm g_j \geq 0 \qquad \text{in } \overline{Q}_T$$

for any $j = 1, \dots, n$ if $c_1 \geq \nu_1^{-1}\max_{Q_T}|g_x|$, and therefore

$$\frac{c_1}{2}(x-x^0)^2 \pm u_j(x,t) \geq \max_{(x,t)\in\partial' Q_T}\left[\frac{c_1}{2}(x-x^0)^2 \pm u_j(x,t)\right]$$

for any $(x,t) \in \overline{Q}_T$.

By (14)–(18), majorants for $\max_{\partial'' Q_T}|u_x|$ can be taken from the work [3] devoted to the Dirichlet problem for the stationary equation (2) with a bounded strictly positive function $\Psi$. In fact, we consider $u(x,t)$ as an admissible solution to the problem

$$\begin{cases} F_m(u_{xx}(x,t)) = \Psi(x,t) \equiv g(x,t) + u_t(x,t), & x \in \Omega, \\ u(x,t) - \varphi(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{19}$$

with $t$ as a parameter. It follows from (14)–(18) that

$$0 < \nu_4 \leq \Psi(x,t) \leq \mu_1, \tag{20}$$

where $\mu_1 = \max_{Q_T} g + \max_{\partial' Q_T} u_t + \frac{b}{2}d^2 < \infty, \nu_4 = \nu_3$ or $\widetilde{\nu}_2$.

To estimate $\max_{Q_T}|u_{xx}|$ we make use of the crucial property of $F_m$ — being a concave function of $(u_{ij})$ into $K_m$. By this property equation $(10_3)$ implies

$$L_u(u_{\gamma\gamma}) \geq g_{\gamma\gamma} \geq -\mu_2 \tag{21}$$

for any $\gamma \in \mathbb{R}^n$ and $\mu_2 = \max\{0; \max_{Q_T}(-g_{\gamma\gamma})\}$. This inequality and (11) give

$$L_u\left(u_{\gamma\gamma} + \frac{c_2}{2}(x-x^0)^2\right) \geq -\mu_2 + c_2\nu_1 \geq 0$$

if $c_2 \geq \mu_2/\nu_1$, and from the Hopf Theorem we conclude that

$$\max_{Q_T} u_{\gamma\gamma} \leq \max_{\partial' Q_T} u_{\gamma\gamma} + \frac{c_2}{2}d^2, \qquad c_2 = \mu_2/\nu_1.$$

In this way estimation of $u_{\gamma\gamma}$ in $Q_T$ from above is reduced to that on $\partial'' Q_T$. An estimate of $u_{\gamma\gamma}$ in $\overline{Q}_T$ from below follows from the estimates of all $u_{\gamma_k\gamma_k}$ from above and from the inequality $F_1(u_{xx}) = \Delta u > 0$, which holds for any $u \in K_m$. The freedom in the choice of $\gamma$ yields the estimates $\max_{Q_T} |u_{ij}| \leq c_3$ for all $i, j = 1, \dots, n$ with some constant under control.

Thus our next aim is to find bounds for $|u_{ij}|$ on $\partial'' Q_T$. Some bounds for the second derivatives $u_{\gamma_k\gamma_l}$ of $u$ in directions $\gamma_k$ and $\gamma_l$ tangent to $\partial\Omega$ follow from the inequalities $\max_{\partial'' Q_T} |u_x| \leq c$ and $\max_{\partial'' Q_T} |\varphi_{xx}| \leq c$. In order to majorize $\max_{\partial'' Q_T} |u_{\gamma_k\gamma_n}|$ where $\gamma_k$ is tangent and $\gamma_n$ normal to $\partial\Omega$, we use $(10_2)$ and an observation from [9], [3]:

$$L_u(x_k u_l - x_l u_k) = x_k g_l - x_l g_k, \qquad k, l = 1, \dots, n,$$

which reflects the rotation invariance of $F_m(u_{xx})$.

Take a point $x^0 \in \partial\Omega$ and the "corresponding" cartesian coordinates $x = (\widetilde{x}, x_n)$. In the cylinder

$$Q_T^\varepsilon(x^0) = \{(x,t) \in Q_T \,:\, |\widetilde{x}| < \varepsilon,\ x_n \in (\omega(\widetilde{x}), \omega(\widetilde{x}) + 2\varepsilon),\ t \in [0,T]\}$$

with a small $\varepsilon \in (0,1]$ we define "barriers" $w_\pm$ for the functions $v^k = T_k(u)$, $k < n$, where $T_k = \partial_k + \omega_{kk}(x^0)(x_k\partial_n - x_n\partial_k)$. The barriers have the form $w_\pm = \mp w + T_k(\varphi)$, where

$$w = A\Big[ - x_n + \frac{B}{2}x_n^2 + \frac{1}{2}\sum_{l=1}^{n-1} \omega_{ll}(x^0)x_l^2 - \delta|\widetilde{x}|^2\Big]$$

with positive constants $A, B$ and $\delta$. We can choose $\delta$ and $\varepsilon$ so small and $A$ and $B$ so large that for any $k < n$,

$$(22_1) \qquad L_u(w_+) \leq L_u(v^k) \leq L_u(w_-), \qquad (x,t) \in Q_T^\varepsilon(x^0),$$

and

$$(22_2) \qquad w_- \leq v^k \leq w_+, \qquad (x,t) \in \partial' Q_T^\varepsilon.$$

By the Hopf Theorem, $(22_i)$, $i = 1, 2$, guarantee the inequalities $w_-(x,t) \leq v^k(x,t) \leq w_+(x,t)$, $(x,t) \in \overline{Q}_T^\varepsilon$, which together with $v^k(0,t) = w_\pm(0,t)$ permit us to calculate a majorant for $|u_{kn}(0,t)|$, $k < n$. Thus we have explained how to find majorants for

$$\max_{\partial'' Q_T} |\partial^2_{\gamma_k\gamma_n} u|, \qquad k < n.$$

A majorant for $\max_{\partial'' Q_T} |\partial^2_{\gamma_n\gamma_n} u|$ can be taken from the work [3] devoted to the Dirichlet problem for equations (2) and some their generalizations. This is due to the fact that in [3] the only information used about $\Psi$ was (20).

Note that the construction of a majorant for $|u_{\gamma_n\gamma_n}|$ in [3] is very complicated and artificial. We have a more "transparent" construction and intend to publish it in another paper.

Let us summarize the results:

Proposition 2. *Under the assumptions of Theorem* 1, *the values* $\max_{Q_T}(|u|, |u_x|, |u_t|, |u_{xx}|)$ *for an admissible solution* $u \in C^{4,2}(\overline{Q}_T)$ *to problem* (1), (3) *can be estimated by constants which are determined by the norm of* $g$ *in* $C^{2,1}(\overline{Q}_T)$, *the norm of* $\varphi$ *in* $C^{4,2}(\overline{Q}_T)$, *constants from* (5) (*or* (6)), *the norm of* $\partial\Omega$ *in* $C^4$ *and by the distance of the matrix* $(\omega_{\alpha\beta}(x^0))$ *to* $\partial K_{m-1}$ (*see condition* $(4_2)$).

## 3. Estimation of the Hölder constants for $u_t$ and $u_{ij}$

Now we intend to estimate the Hölder constants $\langle\cdot\rangle^{(\beta)}_{Q_T}$ with some $\beta > 0$ for $u_t$ and $u_{ij}$ in $\overline{Q}_T$. From Proposition 2 we know the constants $\nu$ and $\mu$ in the inequalities

$$\nu \sum_{n=1}^{n} \xi_i^2 \le F_{ij}\xi_i\xi_j \le \mu \sum_{i=1}^{n} \xi_i^2, \qquad \forall \xi_i \in \mathbb{R}^1,\ 0 < \nu \le \mu. \tag{23}$$

Theorem 1.1 of [18] gives a majorant for $\langle u_t\rangle^{(\beta)}_{Q_T}$ with some $\beta > 0$ since $u_t$ can be considered as a bounded solution of the linear uniformly parabolic equation $(10_1)$ with smooth known initial-boundary values and known $\nu$, $\mu$ and $M \equiv \max_{Q_T}|u_t|$.

Concerning the second spatial derivatives $u_{ij}$ of $u$ we first estimate their Hölder constants on $\partial'' Q_T$ making use of cartesian coordinates $(\widetilde{x}, x_n)$ corresponding to $x^0 \in \partial\Omega$. Majorants for the Lipschitz constants for $u_{kl}$ on $\partial'' Q_T$ with $k, l < n$ follow from the relation

$$u(\widetilde{x}, \omega(\widetilde{x}), t) = \varphi(\widetilde{x}, \omega(\widetilde{x}, t), t), \qquad |\widetilde{x}| < \varepsilon,$$

differentiated twice with respect to $x_k$ and $x_l$. Majorants for $\langle u_{kn}\rangle^{(\beta)}_{\partial'' Q_T}$, $k < n$, with a $\beta > 0$ are derived with the help of Theorem 5.1 of [19] (or [20]). This theorem is applied to the functions $\widetilde{v}^k\zeta$, $k < n$, where $\widetilde{v}^k(x,t) = u_k(x,t) -$

$\varphi_k(x,t) + \omega_k(\widetilde{x})[u_n(x,t) - \varphi_n(x,t)]$ and $\zeta = \zeta(x)$ is a smooth cut-off function which is zero outside the ball $B_\varepsilon(x^0)$, $\varepsilon \ll 1$. Each $\widetilde{v}^k\zeta$ satisfies in $\widetilde{Q}_T^\varepsilon(x^0) = [\Omega \cap B_\varepsilon(x^0)] \times (0,T)$ the equation $L_u(\widetilde{v}^k\zeta) = \Phi_k$ with a known majorant for $\max_{Q_T} |\Phi_k|$, and $\widetilde{v}^k\zeta$ is equal to zero on $\partial''\widetilde{Q}_T^\varepsilon(x^0)$. This information is sufficient to get majorants for $\langle u_{kn}\rangle^{(\beta)}_{\partial''Q_T}$, $k < n$. A majorant for $\langle u_{nn}\rangle^{(\beta)}_{\partial''Q_T}$ is calculated elementarily from (1).

Finally, majorants for $\langle u_{ij}\rangle^{(\beta)}_{Q_T}$ are guaranteed by Theorem 2.

THEOREM 2. *Suppose $u \in C^{4,2}(\overline{Q}_T)$ satisfies the equation*

$$-u_t + F(x,t,u,u_x,u_{xx}) = 0. \tag{24}$$

*Assume that*

(a) $\max_{Q_T} |u| \leq M_0$, $\max_{Q_T} |u_x| \leq M_1$, $\max_{Q_T} |u_{xx}| \leq M_2$, $\langle u_{ij}\rangle^{(\beta)}_{\partial' Q_T} \leq M_3$, $\langle u_t\rangle^{(\beta)}_{Q_T} \leq M_4$, $\beta > 0$;

(b) $F \in C^2(\Pi)$, *where*

$$\Pi = \{(x,t,u,p,r) \in \mathbb{R}^{2+2n+n^2} : (x,t) \in \overline{Q}_T,\ |u| \leq M_0,\ |p| \leq M_1,\ |r| \leq M_2\}$$

*and* $\|F\|_{C^2(\Pi)} \leq M_5$;

(c) *equation* (24) *is parabolic on the solution $u$, i.e.,*

$$\nu\xi^2 \leq \frac{\partial}{\partial u_{ij}} F(x,t,u,u_x,u_{xx})|_{u=u(x,t)}\xi_i\xi_j \leq \mu\xi^2, \qquad 0 < \nu \leq \mu,\ \forall \xi_i \in \mathbb{R}^1;$$

(d) *$F$ is a concave function of $u_{ij}$ on $u = u(x,t)$, i.e.,*

$$\frac{\partial^2}{\partial u_{ij}\partial u_{kl}} F(x,t,u,u_x,u_{xx})|_{u=u(x,t)}\xi_{ij}\xi_{kl} \leq 0, \qquad \forall \xi_{ij} \in \mathbb{R}^1.$$

*Then there exists a $\beta_1 \in (0,\beta]$ such that*

$$\langle u_{ij}\rangle^{(\beta_1)}_{Q_T} \leq \Phi(M_0, M_1, M_2, M_3, M_4, M_5, \nu^{-1}, \mu, n), \tag{25}$$

*where $\Phi$ is a positive nondecreasing continuous function of the indicated arguments. It is assumed that $\partial\Omega$ satisfies condition* (A) *(see* §1, *Ch.* I *of* [17]*). $\Phi$ and $\beta_1$ depend on the constants $a_0$ and $\Theta$ appearing in the condition* (A).

Recall that $C^1$-surfaces satisfy condition (A).

Theorem 2 generalizes the results of N. V. Krylov [16] to the case of non-totally parabolic equations. Theorem 2 is proved in [14]. So the following proposition is true.

PROPOSITION 3. *Under the assumptions of Proposition* 2, *the Hölder norms $\langle u_t\rangle^{(\beta)}_{Q_T}$ and $\langle u_{ij}\rangle^{(\beta)}_{Q_T}$ with some $\beta > 0$ can be estimated by constants determined by the quantities indicated in Proposition* 2.

As was explained in the Introduction, Proposition 3 permits us to use the results on linear parabolic equations to prove Theorem 1.

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