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PERIODIC PROCESSES AND NON-AUTONOMOUS EVOLUTION EQUATIONS WITH TIME-PERIODIC TERMS[1]

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Dedicated to Jean Leray

1. Introduction

We study attractors of periodic processes corresponding to non-autonomous evolution equations with right-hand sides periodic in time. The notion of a process generalizes the notion of a semigroup which describes the dynamics of autonomous equations. We consider Cauchy problems of the type

$$\partial_t u = A(u, t), \quad u|_{t=\tau} = u_\tau, \qquad t \geq \tau,\ \tau \in \mathbb{R}. \tag{1}$$

Here $A(\cdot, t) : E_1 \to E_0$, $t \in \mathbb{R}$, is a family of non-linear operators periodic in time with period $p : A(\cdot, t+p) = A(\cdot, t)$ for $t \in \mathbb{R}$, where E_1 and E_0 are Banach spaces, usually with $E_1 \subseteq E_0$. The initial data u_τ is taken in a Banach space E. Assume that for any $\tau \in \mathbb{R}$ and every $u_\tau \in E$ there exists a unique solution $u(t)$, $t \geq \tau$, of the problem (1) such that $u(t) \in E$ for all $t \geq \tau$. Consider the two-parametric family of mappings $\{U(t, \tau) : t \geq \tau, \tau \in \mathbb{R}\}$, $U(t,\tau) : E \to E$, $U(t,\tau)u_\tau = u(t)$, $t \geq \tau$, where $u(t)$ is the solution of the problem (1). The family $\{U(t,\tau)\}$ is said to be the process corresponding to the problem (1). Evidently, the process $\{U(t,\tau)\}$ is periodic in time with period p, i.e. $U(t+p, \tau+p) = U(t,\tau)$

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for all $t \geq \tau$, $\tau \in \mathbb{R}$. We use the notion of the attractor to describe the limit behavior of the process as $t-\tau$ tends to infinity. The attractor $\mathcal{A}$ of $\{U(t,\tau)\}$ is a minimal closed attracting set of the process. The attracting property of $\mathcal{A}$ means that for any bounded set $B \subseteq E$, $\operatorname{dist}_E(U(t,\tau)B, \mathcal{A}) \to 0$ as $t-\tau \to +\infty$. The property of minimality is the natural generalization of the invariance property in the definition of a semigroup attractor.

One method to construct the attractor for a periodic process is to study the attractors $\mathcal{A}(\delta)$ of the discrete semigroups $\{S_n(\delta)\}_{n\in\mathbb{Z}_+}$ where $S_n(\delta) = U(\delta + np, \delta)$, $\delta \in \mathbb{R}$ (see [9]). The union of these attractors is the attractor of the initial periodic process:

$$\mathcal{A} = \bigcup_{\delta\in\mathbb{R}} \mathcal{A}(\delta).$$

Another way, described in [11] and [12], is the direct investigation of ω-limit sets of the process.

We present an alternative approach. We study a continuous semigroup $\{S(t) : t \geq 0\}$ acting in the extended phase space $E \times \mathbb{T}^1$, where $\mathbb{T}^1$ is the circle of length p. The operators $S(t)$ are defined by

$$S(t)(u,\, \delta) = (U(\delta + t,\, \delta)u,\, (\delta + t) \operatorname{mod} p). \tag{2}$$

It is easily seen that (2) defines a semigroup in $E \times \mathbb{T}^1$.

In Section 2 we formulate the main definitions and theorems on the attractors of general processes from [3], [4] and [6] that we intend to use.

In Section 3 we prove theorems on the existence and structure of the attractor for a periodic process. We also give some properties of the attractor which describe the character of attraction.

In Section 4 we estimate from above the fractal and Hausdorff dimensions of the attractors of periodic processes.

Section 5 contains applications of the above results to problems arising in mathematical physics. We study the following equations and systems:

(i) the two-dimensional Navier-Stokes system with external forces periodic in time;

(ii) non-autonomous reaction-diffusion system with periodic interaction function $f(u,t)$ and with periodic external forces $\varphi(x,t)$ ($f(u,t+p) = f(u,t)$, $\varphi(x,t+p) = \varphi(x,\, t)$);

(iii) damped hyperbolic equation with periodic terms.

Note that the dimension of the attractor $\mathcal{A}$ of a periodic process in all the above examples satisfies $\dim \mathcal{A} \leq \dim \mathcal{A}(0) + 1$, where $\mathcal{A}(0)$ is the attractor of the corresponding discrete semigroup. This inequality was formulated by Haraux in [11] as a conjecture.

Finally, notice that [3], [4] and [6] contain estimates from above for the Hausdorff dimension of the attractors of non-autonomous equations and systems with quasiperiodic terms.

2. Preliminaries

First of all, we recall some definitions concerning processes and their attractors. A two-parametric family of mappings $\{U(t,\tau)\} = \{U(t,\tau) : t \geq \tau,\ \tau \in \mathbb{R}\}$, $U(t,\tau) : E \to E$, acting in a Banach space E is said to be a *process* if

$$U(t,\tau) = U(t,\, s)U(s,\tau), \quad U(\tau,\tau) = I, \qquad \forall \tau \in \mathbb{R},\ \tau \leq s \leq t.$$

A process $\{U(t,\tau)\}$ is called *periodic* with period p if

$$U(t+p,\tau+p) = U(t,\tau), \qquad \forall\, t \geq \tau,\ \tau \in \mathbb{R}.$$

A set $P_0 \subseteq E$ is said to be an *attracting set* of the process $\{U(t,\tau)\}$ if for any $\tau \in \mathbb{R}$ and for every bounded set $B \subseteq E$,

$$\text{dist}_E(U(t,\tau)B,\, P_0) \to 0 \qquad (t \to +\infty). \tag{1}$$

A set $P \subseteq E$ is said to be a *uniformly* (in $\tau \in \mathbb{R}$) *attracting set* of the process $\{U(t,\tau)\}$ if (1) holds uniformly in $\tau \in \mathbb{R}$, i.e.

$$\sup_{\tau \in \mathbb{R}} \text{dist}_E(U(T+\tau,\tau)B,\, P) \to 0 \qquad (T \to +\infty). \tag{2}$$

A process $\{U(t,\tau)\}$ is called *asymptotically* (*uniformly asymptotically*) *compact* if it has a compact attracting (uniformly attracting) set.

DEFINITION 1. (i) A closed set $\mathcal{A}_0 \subseteq E$ is said to be the *attractor* of a process $\{U(t,\tau)\}$ if $\mathcal{A}_0$ is a minimal closed attracting set of $\{U(t,\tau)\}$. The minimality means that any closed attracting set contains $\mathcal{A}_0$.

(ii) A closed set $\mathcal{A}_1 \subseteq E$ is said to be the *uniform* (in $\tau \in \mathbb{R}$) *attractor* of a process $\{U(t,\tau)\}$ if $\mathcal{A}_1$ is a minimal closed uniformly (in $\tau \in \mathbb{R}$) attracting set of $\{U(t,\tau)\}$.

These definitions were introduced in [11] and [12]. To construct the attractor for a periodic process, we shall use the results of [6], where we studied the attractors for more general processes and families of processes. For the sake of completeness, we recall the necessary definitions and theorems from [6].

Suppose we are given a family of processes $\{U_\sigma(t,\tau)\}$, depending on a function parameter σ in a complete metric space Σ. The parameter σ is called the *symbol* of the process $\{U_\sigma(t,\tau)\}$, and Σ is the *symbol space*. By analogy, we introduce the following definitions.

DEFINITION 2. (i) A set P_Σ is said to be *uniformly* (in $\sigma \in \Sigma$) *attracting* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for any $\tau \in \mathbb{R}$ and every bounded set $B \subseteq E$,

$$\sup_{\sigma\in\Sigma} \operatorname{dist}_E(U_\sigma(t,\tau)B,\, P_\Sigma) \to 0 \qquad (t \to +\infty).$$

(ii) A set $\mathcal{A}_\Sigma$ is said to be the *uniform attractor* of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if it is a minimal closed uniformly (in $\sigma \in \Sigma$) attracting set of that family.

A family of processes with a compact uniformly attracting set is called *uniformly asymptotically compact.* As was shown in [6] (and also in [11] and [12] using different terminology), a uniformly asymptotically compact family of processes always has a compact uniform attractor.

Now we shall investigate the structure of uniform attractors more closely under some additional conditions. We assume that some strictly invariant semigroup $\{T(t) : t \geq 0\}$ acts on Σ: $T(t) : \Sigma \to \Sigma$ and $T(t)\Sigma = \Sigma$ for all $t \geq 0$. Let us also assume the following translation identity:

$$U_\sigma(t+s,\tau+s) = U_{T(s)\sigma}(t,\tau), \qquad \forall \sigma \in \Sigma,\ t \geq \tau,\ t,\tau \in \mathbb{R},\ s \geq 0. \tag{3}$$

We define the family $\{S(t) : t \geq 0\}$ of mappings of $E \times \Sigma$ into $E \times \Sigma$ by:

$$S(t)(u,\,\sigma) = (U_\sigma(t,\,0)u,\, T(t)\sigma), \qquad t \geq 0,\ (u,\,\sigma) \in E \times \Sigma. \tag{4}$$

One can easily check using (3) that $\{S(t)\}$ is a semigroup acting on $E \times \Sigma$: $S(t_1)S(t_2) = S(t_1+t_2)$ for all $t_1, t_2 \geq 0$ and $S(0) = I$ (see [6]).

A curve $\{u(s) : s \in \mathbb{R}\}$ in E is said to be a *complete trajectory* of the process $\{U(t,\tau)\}$ if $U(t,\tau)u(\tau) = u(t)$ for all $t \geq \tau$, $t,\tau \in \mathbb{R}$.

DEFINITION 3. The *kernel* $\mathcal{K}$ of the process $\{U(t,\tau)\}$ consists of all bounded complete trajectories of $\{U(t,\tau)\}$:

$$\mathcal{K} = \{u(\cdot) : u(t),\ t \in \mathbb{R},\ \text{is a complete trajectory of } \{U(t,\tau)\} \\ \text{and } \|u(t)\|_E \leq C_u \ \forall t \in \mathbb{R}\}.$$

The set $\mathcal{K}(s) = \{u(s) : u(\cdot) \in \mathcal{K}\}$ is called the *kernel section* at time $t = s$, $s \in \mathbb{R}$.

In the sequel $\Pi_1 : E \times \Sigma \to E$ and $\Pi_2 : E \times \Sigma \to \Sigma$ are the canonical projections.

A family of operators $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is said to be $(E \times \Sigma,\, E)$-*continuous* if for any fixed t and τ the mapping $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ is continuous from $E \times \Sigma$ into E.

Let us formulate the main theorem on attractors of families of processes.

THEOREM 1. *Suppose a family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, *acting in a space* E *is uniformly (in* $\sigma \in \Sigma$*) asymptotically compact and* $(E \times \Sigma, E)$*-continuous. Moreover, suppose* Σ *is a compact metric space and let* $\{T(t)\}$ *be a continuous strictly invariant semigroup on* Σ *satisfying the translation identity* (3). *Then the semigroup* $\{S(t)\}$ *corresponding to the family* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, *and acting on* $E \times \Sigma$ *by the formula* (4) *has a compact attractor* $\mathcal{A} : S(t)\mathcal{A} = \mathcal{A}$ *for all* $t \geq 0$. *Moreover,*

(i) $\Pi_1\mathcal{A} = \mathcal{A}_\Sigma$ *is the uniform (in* $\sigma \in \Sigma$*) attractor of the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$;

(ii) $\Pi_2\mathcal{A} = \Sigma$;

(iii) $\mathcal{A}_\Sigma = \bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(0)$. *Here* $\mathcal{K}_\sigma$ *is the kernel of the process* $\{U_\sigma(t,\tau)\}$ *with symbol* $\sigma \in \Sigma$.

Note that the section $\mathcal{K}_\sigma(0)$ in (iii) can be replaced by any $\mathcal{K}_\sigma(t)$, where $t \in \mathbb{R}$.

Theorem 1 was proved in [6]. It follows from a general theorem on semigroup attractors (see, for example, [1], [10] and [17]) applied to the semigroup (4). Papers [3], [4] and [6] contain many examples of different non-autonomous dynamical systems having uniform attractors according to Theorem 1.

In Section 3 we apply Theorem 1 to the study of the attractors of periodic processes.

3. Attractors of periodic processes

Let $\{U(t,\tau)\}$ be a periodic process (with period p, $p > 0$) acting in a Banach space E: $U(t+p,\tau+p) = U(t,\tau)$ for all $t \geq \tau$; $t,\tau \in \mathbb{R}$. Let $\{U_\sigma(t,\tau)\}$ be the family of processes depending on $\sigma \in \mathbb{T}^1$ ($\mathbb{T}^1 = \mathbb{R} \mod p$ is a one-dimensional torus), defined by

$$U_\sigma(t,\tau) = U(t+\sigma,\tau+\sigma). \tag{1}$$

Clearly, the existence of a uniformly (in $\sigma \in \mathbb{T}^1$) attracting set $P \subseteq E$ for the family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathbb{T}^1$, is equivalent to the existence of a uniformly (in $\tau \in \mathbb{R}$) attracting set for the original periodic process $\{U(t,\tau)\}$. Note that, by periodicity, the uniform (in $\tau \in \mathbb{R}$) attracting property (2.2) is equivalent to the uniform attracting property with respect to $\tau \in [0, p)$:

$$\sup_{\tau\in[0,p)} \operatorname{dist}_E(U(T+\tau,\tau)B,\, P) \to 0 \qquad (T \to +\infty).$$

The following rotation semigroup $\{T(t)\}$ acts on the symbol space $\Sigma = \mathbb{T}^1$:

$$T(t)\sigma = (t+\sigma) \bmod p, \qquad t \geq 0,\, \sigma \in \mathbb{T}^1.$$

Clearly, the translation identity (1.3) is valid. Indeed,

$$\begin{aligned}U_\sigma(t+s,\tau+s) &= U(t+s+\sigma,\tau+s+\sigma)\\ &= U(t+(s+\sigma)(\operatorname{mod}p),\tau+(s+\sigma)(\operatorname{mod}p))\\ &= U(t+T(s)\sigma,\tau+T(s)\sigma) = U_{T(s)\sigma}(t,\tau).\end{aligned}$$

Consequently, the family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathbb{T}^1$, generates the semigroup $\{S(t)\}$ acting in the extended phase space $E \times \mathbb{T}^1$ by the formula (2.4):

$$S(t)(u,\,\sigma) = (U(t+\sigma,\,\sigma)u,\,(t+\sigma) \bmod p)), \qquad t \geq 0,\ (u,\,\sigma) \in E \times \mathbb{T}^1. \tag{2}$$

Let us formulate the theorem on the attractor of a periodic process.

THEOREM 1. *Let $\{U(t,\tau)\}$ be a periodic, uniformly (in $\tau \in \mathbb{R}$) asymptotically compact, and $(E\times\mathbb{T}^1,\,E)$-continuous process. Then the semigroup $\{S(t) : t \geq 0\}$ acting in $E \times \mathbb{T}^1$ by means of the formula (2) has a compact, strictly invariant attractor $\mathcal{A}$: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$. Moreover,*

(i) *$\Pi_1\mathcal{A} = \mathcal{A}_1$ is the uniform (in $\tau \in \mathbb{R}$) attractor of the process $\{U(t,\tau)\}$;*
(ii) *$\mathcal{A}_1 = \bigcup_{\sigma\in[0,\,p)=\mathbb{T}^1} \mathcal{K}(\sigma)$, where $\mathcal{K}(\sigma)$ is the section at time $t = \sigma$ of the kernel $\mathcal{K}$ of the process $\{U(t,\tau)\}$.*

PROOF. This follows from Theorem 2.1.

REMARK 1. Notice that the set $\mathcal{A}_1 = \mathcal{A}_0$ also serves as the (non-uniform) attractor of the periodic process $\{U(t,\tau)\}$. In other words, under the assumptions of Theorem 1, the uniform attractor of the periodic process coincides with the (non-uniform) attractor of this process.

The proof of this assertion is given in [7]. For more general processes it is, in general, not true. A counter-example was constructed by Haraux in [12].

Below we study in more detail the kernel sections $\mathcal{K}(t)$, $t \in \mathbb{R}$, of a periodic process $\{U(t,\tau)\}$ satisfying the assumptions of Theorem 1.

Notice that if $u(\,\cdot\,) \in \mathcal{K}$ then $u_p(\,\cdot\,) \in \mathcal{K}$, where $u_p(t) = u(t+p)$. Therefore,

$$\mathcal{K}(t+p) = \mathcal{K}(t), \qquad \forall t \in \mathbb{R}. \tag{3}$$

PROPOSITION 2. *The following identity is valid:*

$$U(t,\tau)\mathcal{K}(\tau) = \mathcal{K}(t), \qquad t \geq \tau;\ t,\tau \in \mathbb{R}. \tag{4}$$

This follows directly from the definition of kernel sections.

We shall prove below that the attractor $\mathcal{A}_0$ of the periodic process $\{U(t,\tau)\}$ can be obtained in another way using attractors of the corresponding discrete semigroups. Let us introduce the family of discrete semigroups $\{S_n(\delta) : n \in \mathbb{Z}_+\}$ depending on a parameter $\delta \in \mathbb{T}^1$:

$$S_n(\delta) = U(\delta + np, \delta), \qquad S_n(\delta) : E \to E, \qquad n \in \mathbb{Z}_+, \ \delta \in \mathbb{T}^1. \tag{5}$$

For any $\delta \in \mathbb{T}^1$ the operators $\{S_n(\delta) : n \in \mathbb{Z}_+\}$ form a semigroup. Indeed,

$$\begin{aligned} S_n(\delta) &= U(np + \delta, \delta) \\ &= U(np + \delta, (n-1)p + \delta)U((n-1)p + \delta, (n-2)p + \delta) \dots U(p + \delta, \delta) \\ &= (U(p + \delta, \delta))^n = (S_1(\delta))^n. \end{aligned}$$

PROPOSITION 3. *Under the assumptions of Theorem* 1, *the kernel section* $\mathcal{K}(\delta)$ *is the attractor of the discrete semigroup* $\{S_n(\delta)\}$.

PROOF. It follows from the $(E \times \mathbb{T}^1, E)$-continuity of the process $\{U(t,\tau)\}$ that each semigroup $\{S_n(\delta)\}$ is (E, E)-continuous. At the same time, it is asymptotically compact, because the periodic process $\{U(t,\tau)\}$ is uniformly asymptotically compact. These two facts imply that each semigroup $\{S_n(\delta)\}$ has a compact attractor $\mathcal{A}(\delta)$:

$$S_n(\delta)\mathcal{A}(\delta) = \mathcal{A}(\delta), \qquad \forall n \in \mathbb{Z}_+, \ \mathcal{A}(\delta) \Subset E. \tag{6}$$

On the other hand, by (3) and (4),

$$S_n(\delta)\mathcal{K}(\delta) = U(np + \delta, \delta)\mathcal{K}(\delta) = \mathcal{K}(np + \delta) = \mathcal{K}(\delta),$$

i.e. $\mathcal{K}(\delta)$ is bounded and strictly invariant (with respect to $\{S_n(\delta)\}$) set. Hence, by the attracting property, $\mathcal{K}(\delta) \subseteq \mathcal{A}(\delta)$ for all $\delta \in \mathbb{T}^1$. Let us check the reverse inclusion. Let $u_\delta \in \mathcal{A}(\delta)$. We shall construct a bounded complete trajectory $u(t), t \in \mathbb{R}$, of the process $\{U(t,\tau)\}$ such that $u(\delta) = u_\delta$. Put $u(t) = U(t,\delta)u_\delta$ for $t \geq \delta$. Evidently, for $t \geq \delta$ the function $u(t)$ is a trajectory of the process. Let us extend $u(t)$ for $t < \delta$. According to (6) the equation

$$S_1(\delta)u_{\delta-1} = u_\delta$$

has at least one solution $u_{\delta-1} \in \mathcal{A}(\delta)$. Now we put $u(\delta - p) = u_{\delta-1}$ and define $u(t)$ for $\delta - p \leq t < \delta$ by $u(t) = U(t, \delta - p)u_{\delta-1}$. Clearly, $u(t)$ is a trajectory of $\{U(t,\tau)\}$ for $t \geq \delta - p$. Continuing this procedure one can construct the trajectory $u(t)$ for $t \geq \delta - np$ so that $u(\delta - np) \in \mathcal{A}(\delta)$. Letting $n \to +\infty$ we get a complete trajectory of the process. Let us show that this trajectory is bounded. The set $\mathcal{A}(\delta)$ is bounded. The process $\{U(t,\tau)\}$ is uniformly asymptotically compact, therefore, for some compact set $P_1 \Subset E$ there exists $T \geq \delta$ such that $U(t,\delta)\mathcal{A}(\delta) \subseteq B_1$ for $t \geq T$. Here $B_1 = \mathcal{O}_\varepsilon(P_1)$ is the ε-neighborhood of P_1 for some ε. In particular, $u(t+np) = U(t,\delta)u(\delta+np) \subseteq B_1$ for all $t \geq T$ and $n \in \mathbb{Z}$. Therefore, $u(t_1) \in B_1$ for any $t_1 \in \mathbb{R}$, i.e. the trajectory $u(t)$ is bounded. Finally, u_δ belongs to the bounded complete trajectory $u(t)$, $u_\delta = u(\delta) \in \mathcal{K}(\delta)$, and $\mathcal{A}(\delta) \subseteq \mathcal{K}(\delta)$. Hence, $\mathcal{A}(\delta) = \mathcal{K}(\delta)$. □

Now we shall formulate some properties of the kernel sections $\mathcal{K}(\delta)$ under the condition that for any fixed $\tau \in \mathbb{R}$ the periodic process $\{U(t,\tau)\}$ is continuous as a mapping $(u,t) \mapsto U(t,\tau)u$ from $E \times \mathbb{R}$ into E.

THEOREM 4. *Suppose a periodic process $\{U(t,\tau)\}$ satisfies the assumptions of Theorem 1 and it is continuous with respect to $u \in E$ and $t \geq \tau$ for any fixed τ. Then:*

(i) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|t-s| < \delta$ implies*

$$\operatorname{dist}_E(\mathcal{K}(t),\mathcal{K}(s)) \leq \varepsilon; \tag{7}$$

(ii) *for any bounded subset B of E,*

$$\operatorname{dist}_E(U(t,\tau)B,\mathcal{K}(t)) \to 0 \qquad \text{as } t \to +\infty. \tag{8}$$

The proof of Theorem 4 can be found in [7].

REMARK 2. The property (8) can be strengthened in the following way:

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}_E(U(T+\tau,\tau)B,\mathcal{K}(T+\tau)) \to 0 \qquad \text{as } T \to +\infty.$$

A process $\{U(t,\tau)\}$ has the *backward uniqueness property*, if $U(t,\tau)u_1 = U(t,\tau)u_2$ implies that $u_1 = u_2$.

PROPOSITION 5. *If a periodic process $\{U(t,\tau)\}$ satisfies the assumptions of Theorem 1 and has the backward uniqueness property then the mapping $U(t,\tau)$: $\mathcal{K}(\tau) \to \mathcal{K}(t)$ is a homeomorphism.*

This follows directly from Proposition 1.

4. Fractal and Hausdorff dimensions of the attractors of periodic processes

We recall the definitions of the fractal and Hausdorff dimensions of subsets of a Banach space E. Let X be a compact subset of E. We denote by $B_r(x)$ the ball in E of radius r with center at x. Given $d \in \mathbb{R}_+$ and $\varepsilon > 0$, we set

$$\mu_H(X, d, \varepsilon) = \inf \sum r_i^d,$$

where the inf is taken over all coverings of X by balls $B_{r_i}(x_i)$ of radii $r_i \leq \varepsilon$. Let $\mu_H(X, d)$ denote the d-dimensional *Hausdorff measure* of X:

$$\mu_H(X, d) = \lim_{\varepsilon \to 0} \mu_H(X, d, \varepsilon) = \sup_{\varepsilon > 0} \mu_H(X, d, \varepsilon).$$

The quantity $d_H(X) = \inf\{d : \mu_H(X, d) = 0\}$ is the *Hausdorff dimension* of X. By analogy, one introduces the fractal dimension of X. Let $n(\varepsilon, X)$ be the minimal number of balls having radii ε which cover X. The *d-dimensional fractal measure* is defined by

$$\mu_F(X, d) = \limsup_{\varepsilon \to 0} \varepsilon^d n(\varepsilon, X) = \limsup_{\varepsilon \to 0} \mu_F(X, d, \varepsilon).$$

The *fractal dimension* of X is

$$d_F(X) = \inf\{d : \mu_F(X, d) = 0\}.$$

It is clear that $\mu_H(X, d) \leq \mu_F(X, d)$ and $d_H(X) \leq d_F(X)$.

Notice that in the definitions of the Hausdorff and fractal dimensions we may use coverings by balls with centers belonging to X. We shall consider only such coverings.

Let K_0 be a compact subset of E. Consider a mapping $\Phi : K_0 \times [0, T] \to E$ such that $\Phi(x, 0) = x$ for all $x \in K_0$. We assume that Φ satisfies the Lipschitz condition with respect to x and t:

$$\text{(1)} \quad \|\Phi(x_1, t_1) - \Phi(x_2, t_2)\|_E \leq k(\|x_1 - x_2\|_E + |t_1 - t_2|), \qquad \forall x_1, x_2 \in K_0,\ \forall t_1, t_2 \in [0, T].$$

PROPOSITION 1. *Let* $K_t = \Phi(K_0, t)$, $t \in [0, T]$. *Then*

$$d_F(K_t) \leq d_F(K_0), \tag{2}$$

$$d_H(K_t) \leq d_H(K_0). \tag{3}$$

PROOF. Fix $\varepsilon > 0$ and let $K_0 \subseteq \bigcup_{i=1}^N B_\varepsilon(x_i)$ with all $x_i \in K_0$. Then

$$K_t \subseteq \bigcup_{i=1}^N \Phi(B_\varepsilon(x_i), t) \subseteq \bigcup_{i=1}^N B_{\varepsilon k}(\Phi(x_i, t))$$

by (1). Therefore, $\mu_F(K_t, d, \varepsilon k) \leq k^d \mu_F(K_0, d, \varepsilon)$, i.e. $\mu_F(K_t, d) \leq k^d \mu_F(K_0, d)$. The inequality (2) is proved. Using the same reasoning one can establish (3). □

PROPOSITION 2. *Let* $\Xi = \bigcup_{t\in[0,T]} \Phi(K_0, t)$. *Then*

$$d_F(\Xi) \le d_F(K_0) + 1, \tag{4}$$

$$d_H(\Xi) \le d_H(K_0) + 1. \tag{5}$$

PROOF. Fix $\varepsilon > 0$ and cover K_0 by balls with radii $\varepsilon/(2k)$ and centers $x_1, \dots, x_N$. Then for any $t \in [0, T]$ the balls $B_{\varepsilon/2}(\Phi(x_1, t)), \dots, B_{\varepsilon/2}(\Phi(x_N, t))$ cover K_t (by (1)). Now $0 = t_0 < t_1 < \dots < t_n = T$ be a partition of $[0, T]$, where $0 < t_{i+1} - t_i \le \varepsilon/(2k)$. Then $n \le 2Tk/\varepsilon$. Consider the collection of balls $\{B_\varepsilon(\Phi(x_j, t_i)) : j = 1, \dots, N, i = 1, \dots, n\}$. This collection covers Ξ. Indeed $\forall\, \Phi(x,t)\ \exists\, x_j : \|\Phi(x,t) - \Phi(x_j,t)\| < \varepsilon/2$ and $\exists\, t_i : |t - t_i| < \varepsilon/(2k)$; then

$$\begin{aligned}\|\Phi(x,t) - \Phi(x_j,t_i)\| &\le \|\Phi(x,t) - \Phi(x_j,t)\| \\ &\quad + \|\Phi(x_j,t) - \Phi(x_j,t_i)\| < \varepsilon/2 + k\varepsilon/(2k) = \varepsilon.\end{aligned}$$

Hence, $\Phi(x,t) \in B_\varepsilon(\Phi(x_j, t_i))$. Therefore,

$$\mu_F(\Xi, d+1, \varepsilon) \le nN\varepsilon^{d+1} \le 2TkN\varepsilon^d = 2Tk(2k)^d N \left(\frac{\varepsilon}{2k}\right)^d,$$

i.e. $\mu_F(\Xi, d+1, \varepsilon) \le T(2k)^{d+1}\mu_F(K_0, d, \varepsilon/(2k))$, and $\mu_F(\Xi, d+1) \le T(2k)^{d+1} \times \mu_F(K_0, d)$. Finally, $d_F(\Xi) \le d_F(K_0) + 1$. Estimate (4) is proved.

To prove (5), we cover the set K_0 by balls with (maybe) different radii: $\{B_{\varepsilon_j}(x_j)\}$, where $\varepsilon_j \le \varepsilon/(2k)$. Then for any fixed point x_j, we partition the segment $[0, T]$ into intervals of length $\tau_j \le \varepsilon_j/(2k)$ by points $0 = t_0^j < t_1^j < \dots < t_{n_j}^j = T$. Then $n_j \le 2Tk/\varepsilon_j$. The family of balls $\{B_{2k\varepsilon_j}(\Phi(x_j, t_i^j)) : j = 1, \dots, N, i = 1, \dots, n_j\}$ covers Ξ. Let us estimate $\mu_H(\Xi, d+1, \varepsilon)$ from above.

$$\begin{aligned}\mu_H(\Xi, d+1, \varepsilon) &\le \sum_j \sum_{i=1}^{n_j} \varepsilon_j^{d+1} \le \sum_j \frac{2Tk}{\varepsilon_j}\varepsilon_j^{d+1} \\ &= 2kT\sum_j (\varepsilon_j)^d = (2k)^{d+1} T \sum_j \left(\frac{\varepsilon_j}{2k}\right)^d.\end{aligned}$$

Hence

$$\mu_H(\Xi, d+1, \varepsilon) \le (2k)^{d+1} T \mu_H(K_0, d, \varepsilon/(2k)).$$

The last inequality implies

$$\mu_H(\Xi, d+1) \le (2k)^{d+1} T\mu_H(K_0, d) \quad \text{and} \quad d_H(\Xi) \le d_H(K_0) + 1.$$

□

Now we apply the above results to the periodic process $\{U(t,\tau)\}$ under consideration.

THEOREM 3. *Suppose the periodic process $\{U(t,\tau)\}$ satisfies the assumptions of Theorem 3.1, and for any compact set $K \Subset E$ the mapping $(u,t) \mapsto U(t,\tau)u$ from $K \times [\tau, \tau+T]$ into E satisfies the Lipschitz condition* (1). *Then*

(i) $d_F(\mathcal{K}(t)) = d_F(\mathcal{K}(0)), d_H(\mathcal{K}(t)) = d_H(\mathcal{K}(0))$ *for all* $t \in \mathbb{R}$, *where* $\mathcal{K}$ *is the kernel of the process* $\{U(t,\tau)\}$,

(ii) $d_F(\mathcal{A}_0) \le d_F(\mathcal{K}(0)) + 1, d_H(\mathcal{A}_0) \le d_H(\mathcal{K}(0)) + 1$, *where* $\mathcal{A}_0$ *is the attractor of the process* $\{U(t,\tau)\}$.

PROOF. Proposition 3.2 implies that $U(t,0)\mathcal{K}(0) = \mathcal{K}(t)$. Therefore, by Proposition 1, $d_F(\mathcal{K}(t)) \le d_F(\mathcal{K}(0))$. On the other hand, $U(np,t)\mathcal{K}(t) = \mathcal{K}(np) = \mathcal{K}(0)$ (see (3.3)) if $np \ge t$, therefore, $d_F(\mathcal{K}(t)) \ge d_F(\mathcal{K}(0))$, and so $d_F(\mathcal{K}(t)) = d_F(\mathcal{K}(0))$. The same reasoning works for the Hausdorff dimension. Part (ii) follows directly from Proposition 2. □

Finally, we conclude that, under certain assumptions, the finite dimensionality of the attractor of a periodic process follows from the finite dimensionality of the kernel sections or (by Proposition 3.3) from the finite dimensionality of the attractors of the discrete semigroups $\{S_n(\delta)\}$.

5. Evolution equations with periodic terms

Consider equations of the type:

$$\partial_t u = A(u,t), \qquad u|_{t=\tau} = u_\tau, \qquad t \ge \tau,\ \tau \in \mathbb{R}, \tag{1}$$

where for any $t \in \mathbb{R}$ the operator $A(\,\cdot\,,t)$ maps a Banach space E_1 into E_0, where $E_1 \subseteq E_0$. We assume that $A(\,\cdot\,,t)$ is time-periodic with period p: $A(\,\cdot\,,t+p) = A(\,\cdot\,,t)$. The initial conditions u_τ of the problem (1) belong to a Banach space E with $E_1 \subseteq E \subseteq E_0$. We assume that for any $\tau \in \mathbb{R}$ and arbitrary $u_\tau \in E$ the problem (1) has a unique solution $u(t) \in E, t \ge \tau$. The meaning of the expression "$u(t)$ is a solution of (1)" should be clarified in each particular case. Consider the two-parametric family of operators $\{U(t,\tau) : t \ge \tau, \tau \in \mathbb{R}\}, U(t,\tau) : E \to E$, $U(t,\tau)u_\tau = u(t)$, where $u(t)$ is the solution of (1). Evidently, $\{U(t,\tau)\}$ is a periodic process acting in E. The kernel $\mathcal{K}$ of this process consists of all bounded solutions $u(t)$ of (1) defined for all $t \in \mathbb{R}$: $\|u(t)\|_E \le C_u$ for all $t \in \mathbb{R}$.

Now we present some examples of dynamical systems arising in mathematical physics. These equations are particular cases of equations with almost periodic and quasiperiodic terms considered in [3], [4], [6]. In these works we have proved the uniform asymptotic compactness and continuity of the corresponding processes. We shall also use the results from [5] on estimating the dimension of kernel sections of non-autonomous equations.

EXAMPLE 1 (Two-dimensional Navier-Stokes system with periodic external force). Excluding the pressure, the system can be written in the form

$$\partial_t u + Lu + B(u,u) = \varphi, \qquad x = (x_1, x_2) \in \Omega \Subset \mathbb{R}^2, \tag{2}$$

$$L = -\nu\Pi\Delta, \qquad B(u,u) = \Pi\sum_{i=1}^{2} u_i \partial_{x_i} u, \qquad \varphi = \Pi\varphi_0, \; u|_{\partial\Omega} = 0,$$

where $u = (u^1, u^2)$, $\varphi = (\varphi^1, \varphi^2)$ (see [13], [18], [1]). By H (resp. H_1) we denote, as usual, the closure of the set $\mathcal{V}_0 = \{v : v \in (C_0^\infty(\Omega))^2, (\nabla, v) = 0\}$ in the norm $\| \;\|$ ($\| \;\|_1$) of the space $(L_2(\Omega))^2$ ($(H_1(\Omega))^2$), by Π the orthogonal projection of $(L_2(\Omega))^2$ on H and its different extensions. We assume that $\varphi(\,\cdot\,, t) \in C_b(\mathbb{R}, H)$ is periodic in t with period p, and moreover, for any $\tau \in \mathbb{R}$ and $T > 0$ the function $\varphi_t'(\,\cdot\,, t) \in L_2([\tau, \tau + T], H_{-1})$, where $H_{-1} = (H_1)^*$. The initial conditions are posed at $t = \tau$:

$$u|_{t=\tau} = u_\tau, \qquad u_\tau \in H \;\; (E = H). \tag{3}$$

It is well known that the problem (2), (3) is uniquely solvable in the class of functions satisfying $u(t) \in C_b([\tau, +\infty), H) \cap L_2([\tau, \tau+T), H_1)$, for all $T > 0$, and $\partial_t u \in L_2([\tau, \tau + T), H_{-1})$. Thus, the process $\{U(t,\tau) : t \geq \tau\}$, $U(t,\tau)u_\tau = u(t)$, acting on H and corresponding to (2), (3) is defined. The process $\{U(t,\tau)\}$ is uniformly (in $\tau \in \mathbb{R}$) asymptotically compact and $(H \times \mathbb{T}^1, H)$-continuous (see [6]). Therefore, Theorem 3.1 is applicable. In particular, the set $\mathcal{A}_0 = \mathcal{A}_1 = \bigcup_{\sigma\in\mathbb{T}^1} \mathcal{K}(\sigma)$ is the attractor and uniform attractor of the process $\{U(t,\tau)\}$; $\mathcal{K}$ is the kernel of the process. It is easily seen that the mapping $(u,t) \mapsto U(t,\tau)u$ is continuous in $(u,t) \in H \times \mathbb{R}_\tau$ for any fixed $\tau \in \mathbb{R}$. Hence, Theorem 3.4 also applies.

In [5] we proved the following estimate for the Hausdorff dimension of the kernel sections $\mathcal{K}(t)$ of the problem (2), (3):

$$\dim_H K(t) \leq \left\lceil \frac{C}{\nu^2} (M_{-1}(|\varphi|^2))^{1/2} \right\rceil, \qquad \forall t \in \mathbb{R}, \tag{4}$$

where $M_{-1}(|\varphi|^2) = \frac{1}{p}\int_0^p |\varphi(s)|_{-1}^2\, ds$, the constant C does not depend on ν and t. Here and below $\lceil a \rceil = \min\{i \in \mathbb{N} : a < i\}$. Theorem 4.1, when applied to the problem (2), (3), can be formulated as follows.

THEOREM 1. *Suppose $\varphi(x,t)$ satisfies the above conditions. Then*

$$d_H(\mathcal{K}(t)) = d_H(\mathcal{K}(0)), \qquad \forall\, t \in \mathbb{R}, \tag{i}$$

(ii) $$d_H(\mathcal{A}_0) \le d_H(\mathcal{K}(0)) + 1 \le \left\lceil \frac{C}{\nu^2}(M_{-1}(|\varphi|^2))^{1/2} + 1 \right\rceil .$$

Analogous estimates are also valid for the fractal dimension. The complete proof of Theorem 1 will be given in [7]. The main point is the proof of the Lipschitz condition (4.1) for the problem (2), (3).

EXAMPLE 2 (Reaction-diffusion system with periodic terms). Consider the following system:

$$\partial_t u = \nu a \Delta u - f(u,t) + \varphi(x,t), \qquad u|_{\partial\Omega} = 0 \ \left(\text{or } \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0 \right), \tag{5}$$

where $x \in \Omega \Subset \mathbb{R}^n$, $a = \{a_{ij}\}_{i,j=1}^N$ is an $N \times N$-matrix with positive symmetric part $a + a^* \ge \beta^2 I, \beta^2 > 0, f = (f^1, \dots, f^N)$, $\varphi = (\varphi^1, \dots, \varphi^N), u = (u^1, \dots, u^N)$. We assume that $\varphi(\,\cdot\,, t) \in C_b(\mathbb{R}, H)$, $\varphi'_t(\,\cdot\,, t) \in C_b(\mathbb{R}, H_{-1})$, where $H = (L_2(\Omega))^N$ and $f, f'_u, f'_t \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^N)$. The functions f and φ are periodic in t with period p: $f(u, t+p) = f(u,t)$, $\varphi(x, t+p) = \varphi(x,t)$. Also assume the following conditions hold for all $t \in \mathbb{R}$ and $u, v \in \mathbb{R}^N$:

$$\gamma_2 |u|^{p_0} - C_2 \le (f, u) \le \gamma_1 |u|^{p_0} + C_1, \qquad \gamma_i > 0, \ 2 \le p_0 \le 2n/(n-2), \tag{6}$$

$$(f'_u v, v) \ge -C_3(v,v), \quad |f'_u| \le C_4(|u|+1)^{p_0-2}, \quad |f'_t| \le C_5(|u|+1)^s, \quad s \le \frac{n+4}{n-2}.$$

For $n = 2$ the numbers p_0 and s can be arbitrary positive and $p \ge 2$ (for brevity). We also assume that

$$|f(u+z,t) - f(u,t) - f'_u(u,t)z| \le C_6(1 + |u| + |z|)^{p_1} |z|^{1+\gamma_0},$$

where $p_1 < 4/(n-2)$, and γ_0 is positive and sufficiently small. We supplement the system (5) with the initial conditions

$$u|_{t=\tau} = u_\tau, \qquad u_\tau \in H = (L_2(\Omega))^N. \tag{7}$$

Problem (5), (7) has (for all $u_\tau \in H$) a unique solution

$$u(t) \in C_b([\tau, +\infty), H) \cap L_2([\tau, \tau+T], (H_0^1(\Omega))^N), \qquad \forall T \in \mathbb{R}.$$

(See, for example, [1, 2].) Thus, a periodic process $\{U(t,\tau)\}$ acting on $(L_2(\Omega))^N$ corresponds to the problem (5), (7). It was shown in [6] that the process $\{U(t,\tau)\}$ is uniformly (in $\tau \in \mathbb{R}$) asymptotically compact and $(H \times \mathbb{T}^1, H)$-continuous. Thus Theorems 3.2 and 3.4 are applicable to $\{U(t,\tau)\}$. In particular, this process has an attractor $\mathcal{A}_0$. Let us formulate a theorem on the dimension of the kernel sections $\mathcal{K}(t)$ and the attractor $\mathcal{A}_0$ of the problem (5), (7).

THEOREM 2. *Under the above assumptions on f and φ,*

$$\dim_H \mathcal{K}(t) = \dim_H \mathcal{K}(0) \leq \lceil C_0/\nu^{n/2} \rceil, \qquad \forall t \in \mathbb{R}, \tag{8}$$

$$\dim_H \mathcal{A}_0 \leq \lceil C_0/\nu^{n/2} + 1 \rceil. \tag{9}$$

The proof of (8) is given in [5] for the case of general dependence on t of f and φ. To prove (8) and (9), let us check the assumptions of Theorem 4.1 for the mapping $\Phi(u,t) = U(t,\tau)u$. Clearly, it is sufficient to establish the Lipschitz condition (4.1):

$$\begin{aligned} |\Phi(t_2,u_2) - \Phi(t_1,u_1)| &= |U(t_2,\tau)u_2 - U(t_1,\tau)u_1| \\ &= |u_2(t_2) - u_1(t_1)| \\ &\leq |u_2(t_2) - u_2(t_1)| + |u_2(t_1) - u_2(t_1)| \\ &\leq k(|u_2(\tau) - u_1(\tau)| + |t_2 - t_1|), \end{aligned} \tag{10}$$

where $u_1(\cdot), u_2(\cdot) \in \mathcal{K}$ and $k = k(|t_2 - t_1|, \nu)$ does not depend on u. Here and in the sequel $|u| = \|u\|_H$.

We shall prove that for any complete trajectory $u(\cdot) \in \mathcal{K}$ we have

$$|\partial_t u(t)| \leq M, \tag{11}$$

where M does not depend on $u(\cdot) \in \mathcal{K}$. Clearly, (11) implies that

$$|u_2(t_2) - u_2(t_1)| \leq M|t_2 - t_1|. \tag{12}$$

From the results of [6] and [1] it follows that for any $u_1(\cdot), u_2(\cdot) \in \mathcal{K}$,

$$|u_2(t_1) - u_1(t_1)| \leq M_1|u_2(\tau) - u_1(\tau)|, \qquad M_1 = M_1(|t_1 - \tau|, \nu). \tag{12'}$$

Obviously, (12) and (12′) give (10).

Let us verify (11). Conditions (6) provide, for any $u(\cdot) \in \mathcal{K}$ (see [1], [6])

$$|u(t)|_1 \equiv \|u(t)\|_{H_1} \leq M_2, \qquad t \in \mathbb{R}, \quad \int_\tau^T |u(\tau_1)|_2^2 \, d\tau_1 \leq C(|T - \tau|), \tag{13}$$

where $|u(t)|_2 = \|u(t)\|_{H_2}$, M_2 does not depend u and t, C is a function of $|T-\tau|$. By taking the scalar product in H of the equation (5) with $\partial_t u$, we obtain, using elementary transformations,

$$|\partial_t u|^2 + \gamma\nu(\nabla u, \nabla \partial_t u) \leq |f(u,t)|^2 + \frac{1}{4}|\partial_t u|^2 + |\varphi|^2 + \frac{1}{4}|\partial_t u|^2. \tag{14}$$

By (6),

$$|f(u,t)|^2 \le C(1 + \|u\|_{L_{2r}}^{2r}), \qquad r = \frac{n+2}{n-2} \qquad (n > 2). \tag{15}$$

According to the Gagliardo-Nirenberg inequality,

$$\|u\|_{L_{2r}} \le C|u|_2^{\vartheta}|u|_1^{1-\vartheta}, \qquad \vartheta = \frac{n}{2} - \frac{n}{2r} - 1.$$

This implies

$$\|u\|_{L_{2r}}^{2r} \le C^{2r}|u|_2^{2r\vartheta}|u|_1^{2r(1-\vartheta)}, \qquad 2r\vartheta = 2. \tag{16}$$

Since $|u|_1 \le M_2$ (see (13)), by (16) and (15),

$$|f(u,t)|^2 = \|f(u,t)\|_H^2 \le C_7\left(1 + \|u\|_{L_{2r}}^{2r}\right) \le C_8\left(1 + |u|_2^2\right). \tag{17}$$

From (17) and (14) we deduce

$$\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}\gamma\nu|\nabla u|^2 \le C_8\left(1 + |u|_2^2\right) + |\varphi|^2.$$

Hence, integrating with respect to t, we have

$$\gamma\nu|u(T)|_1^2 + \int_\tau^T |\partial_t u|^2\,d\tau_1 \le \gamma\nu|u(\tau)|_1^2 + 2C_8\left(T - \tau + \int_\tau^T |u|_2^2\,d\tau_1\right) + 2\int_\tau^T |\varphi|^2\,d\tau_1,$$

and by (13)

$$\int_\tau^T |\partial_t u|^2\,d\tau_1 \le C_5(|T - \tau|). \tag{18}$$

Let $\psi(t) \in C^1(\mathbb{R}, \mathbb{R})$, $\psi(t) \ge 0$, $\psi(t) = 0$ for $t \le \tau$, and $\psi(t) \equiv 1$ for $t \ge \tau + \delta$, $\delta \ll T - \tau$. (In the sequel, $T - \tau = p + \delta$, where p is the period of the process $\{U(t,\tau)\}$.) Differentiating equation (5) in t and taking the scalar product with $\psi(t)\partial_t u$, we get

$$\begin{aligned}
\frac{1}{2}\partial_t(\psi|\,\partial_t u|^2) - \frac{1}{2}\psi_t'|\partial_t u|^2 + \psi|\partial_t u|_1^2
&\le -\psi(f_u(u,t)\partial_t u, \partial_t u) - \psi(f_t(u,t), \partial_t u) + \psi(\partial_t\varphi, \partial_t u) \\
&\le \psi C_3|\partial_t u|^2 + \psi|f_t|_{-1}|\partial_t u|_1 + \psi|\partial_t\varphi|_{-1}|\partial_t u|_1 \\
&\le C_3\psi|\partial_t u|^2 + C_{10}\psi\|f_t\|_{L_q}^2 + \frac{1}{4}\psi|\partial_t u|_1^2 \\
&\quad + \psi|\partial_t\varphi|_{-1}^2 + \frac{1}{4}\psi|\partial_t u|_1^2, \qquad q = \frac{2n}{n+2}.
\end{aligned} \tag{19}$$

Here we have used the following embedding reasoning: $H_1 \subset L_p$ (where $p = 2n/(n-2)) \Rightarrow H_{-1} = (H_1)^* \supset L_q$. By (6) we have

$$
(20) \qquad \begin{aligned} |f_t| &\le C_5\,(1+|u|^s)\,, \\ |f_t|_{-1}^2 &\le C_{11}\|f_t\|_{L_q}^2 = C_{11}\left(\int_\Omega |f_t|^q\,dx\right)^{2s/(qs)} \le C_{12}(1+\|u\|_{L_{sq}}^{2s}), \end{aligned}
$$

for $s = (n+4)/(n-2)$. The Gagliardo-Nirenberg inequality implies

$$
(21) \qquad \begin{aligned} \|u\|_{L_{sq}} &\le C|u|_2^{\vartheta}|u|_1^{1-\vartheta}, \qquad \vartheta = \frac{n}{2} - \frac{n}{sq} - 1, \\ \|u\|_{L_{sq}}^{2s} &\le C^{2s}|u|_2^{2s\vartheta}|u|_1^{2s(1-\vartheta)} \le C_{13}|u|_2^2. \end{aligned}
$$

Here we have used the boundedness of $|u|_1$ for $u(\,\cdot\,) \in \mathcal{K}$. It follows from (19), (20), and (21) that

$$
\frac{1}{2}\partial_t\left(\psi|\partial_t u|^2\right) + \frac{1}{2}\psi|\partial_t u|_1^2 \le C_{14}|\partial_t u|^2 + C_{15}|u|_2^2 + |\varphi|_{-1}^2.
$$

Finally, integrating over t, we obtain, using (18)

$$
\frac{1}{2}|\partial_t u(t)|^2 \le C_{14}\int_\tau^T |\partial_t u|^2 d\tau_1 + C_{15}\int_\tau^T |u|_2^2 d\tau_1 + \int_\tau^T |\varphi|_{-1}^2 d\tau_1 \le C_{16}(|T-\tau|),
$$

for $T \ge t \ge \tau + \delta$. The estimate (11) is established. Theorem 2 is completely proved. □

EXAMPLE 3 (Dissipative hyperbolic equation with periodic terms). Consider the equation

$$
(22) \qquad \partial_t^2 u + \gamma\partial_t u = \Delta u - f(u,t) + \varphi(x,t), \qquad u|_{\partial\Omega} = 0,\ x \in \Omega \Subset \mathbb{R}^3,
$$

where $f(u,t)$ and $\varphi(x,t)$ are periodic in t with period p. Under conditions (see [6], [7]), Theorems 3.1 and 3.4 are applicable. Using the estimate for the Hausdorff dimension of the kernel sections of the equation (22) (see [5]), we can estimate the dimension of the attractor $\mathcal{A}_0$ of the process $\{U(t,\tau)\}$ corresponding to (22) (see [7]).

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