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ON BIFURCATION OF PERIODIC SOLUTIONS FOR ANALYTIC FAMILIES OF VECTOR FIELDS

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Dedicated to Jean Leray

Let $F(\mu; x, y) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be a real analytic mapping defined in a neighbourhood of the origin such that $F(0; 0, 0) = (0, 0)$. Set $F^\mu(x, y) = F(\mu; x, y) : \mathbb{R}^2 \to \mathbb{R}^2$, where $\mu \in \mathbb{R}$. Clearly $F^0(0, 0) = (0, 0)$. We then say that F^μ is an *analytic family of vector fields.*

In this paper we study the problem of bifurcation of a periodic solution from the equilibrium at the origin. The most famous fact concerning this problem is the Hopf bifurcation theorem. There is a lot of versions of this theorem (see [2] for references). All of them need some assumptions about the way the eigenvalues cross the imaginary axis. Our approach is quite different. We investigate two mappings $G, H : (\mathbb{R} \times \mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ defined in terms of F. If $\mathbf{0} \in \mathbb{R} \times \mathbb{R}^2$ is isolated in $G^{-1}(\mathbf{0}), H^{-1}(\mathbf{0})$ then we can calculate the local topological degrees $\deg(G)$, $\deg(H)$ at $\mathbf{0} \in \mathbb{R} \times \mathbb{R}^2$. Theorem 2.1 says that if $\deg(G) = \deg(H) = 1$ and if the field F^0 is internally transversal to every small circle centred at $(0, 0)$ then for every $\mu \neq 0$ sufficiently close to zero there is a non-trivial periodic solution of the system $(\dot{x}, \dot{y}) = F^\mu(x, y)$ lying in a small disc centred at $(0, 0)$. Note that the Hopf theorem cannot be applied in this case (see Remark 2.2).

Our proof is based upon some recent results concerning the number of branches of one-dimensional semianalytic sets, proved by Fukuda et al. [3, 4], Arnold [1], Wall [8] and by the second author [5, 6]. This is why we have to assume that F is analytic. It seems that the method presented here generalizes to the case where F is a C^r-mapping, $r \leq \infty$, and satisfies some generic conditions.

The paper is organized as follows. In Section 1 we recall some facts concerning one-dimensional semianalytic sets. In Section 2 we prove the main theorem. In Section 3 we present examples of concrete calculations. In that section we have used a computer program written by the first author for calculating local topological degrees. One can find its brief description in [5].

1. Preliminaries

Let us introduce the necessary notation. Let $\omega : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}, 0)$ be an analytic function, and let $\nabla\omega = (\partial\omega/\partial x, \partial\omega/\partial y)$ denote the gradient of ω. From now on we assume that $\omega > 0$ and $\nabla\omega \neq 0$ everywhere except at the origin, and $\lim \omega(x,y) = +\infty$ as $\|(x,y)\| \to \infty$. For any $\varepsilon > 0$ let $D_\varepsilon = \{(x,y) \in \mathbb{R}^2 : \omega(x,y) < \varepsilon^2\}$, $S_\varepsilon = \partial D_\varepsilon$, $\overline{D}_\varepsilon = D_\varepsilon \cup S_\varepsilon$. Hence $\overline{D}_\varepsilon$ is a 2-dimensional manifold with a boundary S_ε and $(0,0) \in D_\varepsilon$. Note that the sets D_ε, $\varepsilon > 0$, form an open neighbourhood base of $\mathbf{0}$ and the pair $(\overline{D}_\varepsilon, S_\varepsilon)$ is homeomorphic to $(\overline{D}^2_\varepsilon, S^1_\varepsilon)$, where $\overline{D}^2_\varepsilon$ (S^1_ε resp.) is the closed 2-dimensional disc (circle resp.) of radius ε centred at $\mathbf{0}$. Let $B_\varepsilon = \{(\mu; x, y) \in \mathbb{R} \times \mathbb{R}^2 : \mu^2 + x^2 + y^2 < \varepsilon^2\}$.

Let Ω be a bounded open set in $\mathbb{R}^n$ and let $H : \overline{\Omega} \to \mathbb{R}^n$ be continuous. If $H \neq 0$ on $\partial\Omega$ then $\deg(H, \Omega, \mathbf{0})$ denotes the *topological degree* of H with respect to $\mathbf{0}$ and Ω. If $H : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ is a continuous mapping having an isolated zero at the origin then $\deg(H)$ denotes the *local topological degree at* $\mathbf{0}$, i.e. $\deg(H, \Omega, \mathbf{0})$, where Ω is a neighbourhood of $\mathbf{0}$ such that $\overline{\Omega} \cap H^{-1}(\mathbf{0}) = \{\mathbf{0}\}$.

Let $F = (F_1, F_2) : (\mathbb{R} \times \mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ be an analytic mapping defined in a neighbourhood of the origin. Set $F^\mu(x,y) = F(\mu; x, y)$. Thus $F^\mu : \mathbb{R}^2 \to \mathbb{R}^2$ is a family of vector fields with a parameter $\mu \in \mathbb{R}$. We denote by $J(\mu; x, y)$ the Jacobian $\partial(F_1, F_2)/\partial(x,y)$ at $(\mu; x, y)$. From now on we assume that

(1.1) $\qquad F^0$ has an isolated zero at $(0,0)$,

(1.2) $\qquad \operatorname{rank}[DF(\mathbf{0})] \leq 1$, where DF is the derivative matrix of F.

Let $X = F^{-1}(\mathbf{0})$. From (1.1), $X \cap \{0\} \times U = \{\mathbf{0}\}$ for some neighbourhood $U \subset \mathbb{R}^2$ of the origin. We say that X has an *isolated singular point* at $\mathbf{0}$ if $\mathbf{0}$ is isolated in $\{(\mu; x, y) \in X : \operatorname{rank}[DF(\mu; x, y)] \leq 1\}$. Let $g = g(\mu; x, y)$ be an analytic function vanishing at the origin, let $\Delta = \partial(g, F_1, F_2)/\partial(\mu, x, y)$, and let $G = (\Delta, F_1, F_2) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^3$. Since $\operatorname{rank}[DF(\mathbf{0})] \leq 1$ we have $\Delta(\mathbf{0}) = 0$, and then $G(\mathbf{0}) = \mathbf{0}$. Let $H = (\mu J, F_1, F_2) : (\mathbb{R} \times \mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$. We have (see [7, Lemma 1])

Proposition 1.1.

(i) *If* $\mathbf{0} \in \mathbb{R} \times \mathbb{R}^2$ *is isolated in either* $G^{-1}(\mathbf{0})$ *or* $H^{-1}(\mathbf{0})$ *then* X *has an isolated singular point at* $\mathbf{0}$*, and then* $X \cap B_\varepsilon \setminus \{\mathbf{0}\}$*, for* ε *small enough,* *is homeomorphic to a finite disjoint union of open segments emanating from the origin.*

(ii) *If* $\mathbf{0}$ *is isolated in* $G^{-1}(\mathbf{0})$ *then* $\mathbf{0}$ *is also isolated in* $\{(\mu;x,y) \in X : g(\mu;x,y) = 0\}$*, and then we may assume that* g *has a constant sign on each component of* $X \cap B_\varepsilon \setminus \{\mathbf{0}\}$.

In the above situation, b will denote the number of components of $X \cap B_\varepsilon \setminus \{\mathbf{0}\}$, and b_+ (b_- resp.) the number of components of $X \cap B_\varepsilon \setminus \{\mathbf{0}\}$ on which g is positive (negative resp.). The numbers b, b_+, b_- do not depend on ε for $\varepsilon > 0$ small enough. Moreover, $b = b_+ + b_-$.

We have (see [7, Theorem 3])

THEOREM 1.2. *Assume that* $\mathbf{0}$ *is isolated in* $G^{-1}(\mathbf{0})$ *and* $H^{-1}(\mathbf{0})$*. Then* $b_+ - b_- = 2\deg(G)$, $b = b_+ + b_- = 2\deg(H)$*. In particular,* $b_+ = \deg(G) + \deg(H)$, $b_- = \deg(H) - \deg(G)$.

The above formula for b was proved by Fukuda et al. [4]. A similar formula was also proved in [3]. The formula for $b_+ - b_-$ was proved in [6].

Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}, 0)$ be real analytic, and let $\nabla f = (\partial f/\partial x, \partial f/\partial y) : \mathbb{R}^2 \to \mathbb{R}^2$. Clearly, f has an isolated critical point at $\mathbf{0}$ if and only if $\mathbf{0}$ is isolated in $\nabla f^{-1}(\mathbf{0})$.

PROPOSITION 1.3. *Assume that* f *has an isolated critical point* $\mathbf{0}$*. There is* $\varepsilon > 0$ *such that* $f \neq 0$ *in* $D_\varepsilon \setminus \{\mathbf{0}\}$ *if and only if* $\deg(\nabla f) = 1$.

PROOF. Let $A_- = \{(x,y) \in S^1_\varepsilon : f(x,y) \le 0\}$, where ε is small. Since f is analytic, the set $\{(x,y) \in \overline{D}^2_\varepsilon : f(x,y) \le 0\}$ is homeomorphic to the cone over A_-. Since f has an isolated critical point at $\mathbf{0}$, A_- is either void or S^1_ε or is homeomorphic to a finite union of closed segments. According to [1, 8], the Euler characteristic $\chi(A_-) = 1 - \deg(\nabla F)$. Thus A_- is either void or S^1_ε if and only if $\deg(\nabla f) = 1$.

2. Bifurcation of a periodic solution

Now we can formulate our main result.

THEOREM 2.1. *Let* F^μ *be an analytic family of vector fields satisfying conditions* (1.1) *and* (1.2)*. Let* $F^0 = (F^0_1, F^0_2)$*, let* $f = F^0_1 \partial\omega/\partial x + F^0_2 \partial\omega/\partial y$*, and let* $g = \partial F_1/\partial x + \partial F_2/\partial y$*. Assume that*

(i) *there is* $\varepsilon > 0$ *such that* $f < 0$ ($f > 0$ *resp.*) *in* $\overline{D}_\varepsilon \setminus \{\mathbf{0}\}$,

(ii) $\mathbf{0} \in \mathbb{R} \times \mathbb{R}^2$ *is isolated in* $H^{-1}(\mathbf{0})$ *and* $\deg(H) = 1$,

(iii) *either* $g(0;0,0)$ *is positive* (*negative resp.*) *or*

(iii′) $g(0;0,0) = 0$, $\mathbf{0} \in \mathbb{R} \times \mathbb{R}^2$ *is isolated in* $G^{-1}(\mathbf{0})$ *and* $\deg(G) = 1$ ($\deg(G) = -1$ *resp.*).

Then there is $\delta > 0$ such that for each μ with $0 < |\mu| < \delta$, there is a non-trivial periodic solution of the system $(\dot{x}, \dot{y}) = F^\mu(x, y)$ lying in D_ε.

Proof. Since $\mathbf{0}$ is isolated in $H^{-1}(\mathbf{0})$, it follows from Proposition 1.1(i) that $X = F^{-1}(\mathbf{0})$ has an isolated singular point at $\mathbf{0}$. Hence, from Theorem 1.2, if $\varepsilon > 0$ is small enough then $X \cap B_\varepsilon \setminus \{\mathbf{0}\}$ is the disjoint union of two connected components $X_1 \cup X_2$ emanating from the origin. Clearly, $H^{-1}(\mathbf{0}) = \{(\mu; x, y) \in X : \mu J = 0\}$, thus we may assume that $J \neq 0$ in $X \cap B_\varepsilon \setminus \{\mathbf{0}\}$. So there is $\delta > 0$ such that for each μ with $0 < |\mu| < \delta$,

(1) $X \cap B_\varepsilon \setminus \{\mathbf{0}\} = X_1 \cup X_2$ is transversal to $\{\mu\} \times D_\varepsilon$, and then $\deg(F^\mu, D_\varepsilon, \mathbf{0}) = \sum \operatorname{sign} J(\mu; x, y)$, where $(\mu; x, y) \in X \cap \{\mu\} \times D_\varepsilon$ and the sum consists of at most two elements.

We have assumed that $f < 0$ in $D_\varepsilon \setminus \{\mathbf{0}\}$. Since $f(x, y)$ is equal to the inner product of the vectors $\nabla\omega(x, y)$ and $F^0(x, y)$, for every $(x, y) \in S_\varepsilon$ the vector $F^0(x, y)$ is internally transversal to S_ε. So $\deg(F^0, D_\varepsilon, \mathbf{0}) = 1$ and if $\delta > 0$ is small and $|\mu| < \delta$ then

(2) for every $(x, y) \in S_\varepsilon$ the vector $F^\mu(x, y)$ is internally transversal to S_ε.

In particular, if $|\mu| < \delta$ then

(3) $\deg(F^\mu, D_\varepsilon, \mathbf{0}) = 1$.

From (i), $(X \cap B_\varepsilon \setminus \{\mathbf{0}\}) \cap \{0\} \times D_\varepsilon$ is void and thus from (1), (3) we have

(4) if $0 < |\mu| < \delta$ then $X \cap \{\mu\} \times D_\varepsilon$ consists of one element $(\mu; x(\mu), y(\mu))$ such that $\operatorname{sign} J(\mu; x(\mu), y(\mu)) = 1$.

We may assume that $X_1 = \{(\mu; x(\mu), y(\mu)) : -\delta < \mu < 0\}$, $X_2 = \{(\mu; x(\mu), y(\mu)) : 0 < \mu < \delta\}$. Let $\lambda_i = \lambda_i(\mu)$, $i = 1, 2$, be the eigenvalues of the derivative matrix $[DF^\mu]$ at $(x(\mu), y(\mu))$. Hence $\lambda_1\lambda_2 = J(\mu; x(\mu), y(\mu)) > 0$, $\lambda_1 + \lambda_2 = g(\mu; x(\mu), y(\mu))$. If $g(0; 0, 0)$ is positive then we may assume that $\lambda_1 + \lambda_2$ is positive for $|\mu| < \delta$. If $g(0; 0, 0) = 0$ and $\deg(G) = 1$ then, according to Theorem 1.2, $b_+ = 2$, $b_- = 0$, and thus $\lambda_1 + \lambda_2$ is positive for $0 < |\mu| < \delta$. Hence $\operatorname{Re}(\lambda_i) > 0$, $i = 1, 2$, and therefore, if $0 < |\mu| < \delta$ then the vector field F^μ has a unique zero $(x(\mu), y(\mu)) \in D_\varepsilon$ which repels every orbit starting near it. From (2), no orbit can leave D_ε and thus, according to the Poincaré-Bendixson theorem, there is a non-trivial periodic solution lying in D_ε. The proof of the second version of the theorem is similar.

Remark 2.2. Inspecting the proof shows that we cannot apply the Hopf theorem in this case because the eigenvalues $\lambda_i(\mu)$ do not cross the imaginary axis.

Corollary 2.3. *If a family of analytic vector fields $\overline{F}^\mu = F^{\mu^2}$ ($F^{-\mu^2}$ resp.) satisfies all assumptions of Theorem 2.1 then for every $\varepsilon > 0$ there is $\delta > 0$ such* *that for every μ with $0 < \mu < \delta$ ($-\delta < \mu < 0$ resp.) there is a non-trivial periodic solution of the system $(\dot{x}, \dot{y}) = F^\mu(x, y)$ lying in D_ε.*

3. Examples

In order to illustrate the method we present two examples.

EXAMPLE 3.1. Let

$$F(\mu; x, y) = (-x^3 + \mu y + \mu^4 x, \mu^5 - y^3 - \mu x + \mu^4 y)$$

and

$$\omega = x^2 + y^2.$$

Then

$$F^0 = (-x^3, -y^3), \qquad \nabla\omega = (2x, 2y),$$
$$f = -2x^4 - 2y^4, \qquad g = 2\mu^4 - 3(x^2 + y^2).$$

Clearly $f < 0$ in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. Computer calculations give $\deg(G) = \deg(H) = 1$ and thus, according to Theorem 2.1, non-trivial periodic solutions bifurcate from the equilibrium at $\mathbf{0}$.

EXAMPLE 3.2. Let

$$F(\mu; x, y) = (y^3 - x^5 + \mu y + \mu^3 x + \mu^7 + x^5 y^2, -x^5 - y^5 - \mu x + \mu^4 y - \mu^6 - x^2 y^8)$$

and

$$\omega(x, y) = x^6/6 + y^4/4.$$

Then

$$F^0 = (y^3 - x^5 + x^5 y^2, -x^5 - y^5 - x^2 y^8),$$
$$\nabla\omega = (x^5, y^3),$$
$$f = -x^{10} + x^{10} y^2 - y^8 - x^2 y^{11},$$
$$\nabla f = (-10x^9 + 10x^9 y^2 - 2xy^{11}, 2x^{10} y - 8y^7 - 11x^2 y^{10}).$$

Computer calculations show that $\mathbf{0}$ is isolated in $\nabla f^{-1}(\mathbf{0})$ and $\deg(\nabla f) = 1$. Since $f(x, 0) = -x^{10} < 0$, by Proposition 1.3 there is $\varepsilon > 0$ such that $f < 0$ in $D_\varepsilon - \{\mathbf{0}\}$. We have

$$g = -5x^4 + \mu^3 + 5x^4 y^2 - 5y^4 + \mu^4 - 8x^2 y^7.$$

Computer calculations give $\deg(H) = 3$, so we cannot apply Theorem 2.1. Set $\overline{F}^\mu = F^{\mu^2}$, and let $\overline{G}, \overline{H}$ be the corresponding mappings. Then $\deg(\overline{G}) = \deg(\overline{H}) = 1$, and thus, by Corollary 2.3, non-trivial periodic solutions bifurcate from the equilibrium for $\mu > 0$.

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