SOLUTION SETS AND BOUNDARY
VALUE PROBLEMS IN BANACH SPACES

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(Submitted by L. Górniewicz)

Dedicated to the memory of Juliusz Schauder

1. Introduction

In this paper we present an extension to an arbitrary Banach space $X$ of some
results (Theorem 1.2 and Proposition 1.2) contained in [2] with $X = \mathbb{R}^n$, that is we
deal with the problem

(BV) $\begin{cases}
  \dot{x} = f(t, x) & t \in [a, b] = I \subset \mathbb{R},
  \quad x \in X, \\
x \in S
\end{cases}$

where $X$ is a Banach space, $f : I \times X \to X$ is a continuous map and $S$ is a
subset of the Banach space $C(I, X)$ of continuous functions from $I$ to $X$ with
the maximum norm. The extension is obtained in a quite natural way by using
condensing operators and the related fixed point theory. We look for solution of
(BV) in the form of fixed points of a finite valued upper semicontinuous multivalued
map $\Sigma$, that is the solution map of a suitable “linearized” problem associated to
(BV) (see e.g. [3], [4], [5], [6]). So, in this work, we give an existence result for

Work partially supported by M.U.R.S.T. project (40%) and a C.N.R. bilateral project.

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(BV) as well as for the second order boundary value problem

\[
\begin{aligned}
  x'' &= f(t, x, x', x'') \\
  x &\in S
\end{aligned}
\quad t \in I \subset \mathbb{R}, \quad x \in X,
\]

generalizing Proposition 1.2 in [2]. An example will be given in order to see how this latter result can be applied.

2. Definitions

**Definition 1.1.** Let $X$ and $Y$ be metric spaces. A set valued map $\Sigma : X \to Y$, with nonempty values, is said to be upper semicontinuous at $x \in X$ if for any neighborhood $V$ of $\Sigma(x)$ there exists a neighborhood $U$ of $X$ such that $\Sigma(x) \subset V$ for any $x \in U$. If for every $x \in X$, $\Sigma$ is upper semicontinuous at $x$, then $\Sigma$ is said to be upper semicontinuous (u.s.c.) on $X$. It is well known that $\Sigma$ is u.s.c. if and only if for any closed subset $D \subset Y$ the set $\Sigma^{-1}(D) = \{x \in X : \Sigma(x) \cap D \neq \emptyset\}$ is closed in $X$.

**Definition 1.2.** Let $X$ and $Y$ be topological Hausdorff spaces. A finite valued upper semicontinuous map $\Sigma : X \to Y$ will be called a weighted map (shortly w-map) if, to each $x$ and $y \in \Sigma(x)$, a multiplicity of weight $m(y, \Sigma(x)) \in \mathbb{Z}$ is assigned in such a way that the following property holds: If $U$ is an open set in $Y$ with $\partial U \cap \Sigma(x) = \emptyset$, then

\[
\sum_{y \in \Sigma(x) \cap U} m(y, \Sigma(x)) = \sum_{y' \in \Sigma(x') \cap U} m(y, \Sigma(x'))
\]

whenever $x'$ is close enough to $x$ (see [6], [7]).

**Definition 1.3.** The number $i(\Sigma(x), U) = \sum_{y \in \Sigma(x) \cap U} m(y, \Sigma(x))$ will be called the “index” of multiplicity of $\Sigma(x)$ in $U$. If $U$ is a connected set, the number $i(\Sigma(x))$ does not depend on $x \in X$. In this case the number $i(\Sigma) = i(\Sigma(x), U)$ will be called the index of the weighted map $\Sigma$.

**Definition 1.4.** Let $X$ be a Banach space, $(A, \geq)$ be a partially ordered set. A function $\psi : 2^X \to A$ is said to be a measure of non compactness (MNC) if

\[
\psi(\overline{co}\Omega) = \psi(\Omega) \quad \text{for every } \Omega \in 2^X.
\]

A measure of non compactness is called monotone if $\Omega_0, \Omega_1 \in 2^X$ and $\Omega_0 \subset \Omega_1$ imply $\psi(\Omega_0) \leq \psi(\Omega_1)$.

A real valued MNC $\psi : 2^X \to [0, +\infty)$ is called regular if $\psi(\emptyset) = 0$ is equivalent to the relative compactness of $\Omega$. 
Well known examples of MNC monotone and regular are the following (see [1]):
(a) the Kuratowski MNC defined by \( \alpha(\Omega) = \inf\{d > 0 : \Omega \text{ admits a partition into a finite number of sets whose diameters are less than } d\} \);
(b) the Hausdorff MNC defined by \( \beta(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has finite } \varepsilon \text{ net}\} \). In the following \( \psi \) will be a real valued MNC.

**Definition 1.5.** Let \( X \) be a Banach space and let \( \psi \) be a MNC. A continuous map \( f : \text{dom}(f) \subset X \to X \) is said to be \( \psi \) condensing if there exists \( 0 < h < 1 \) such that
\[
\psi(f(\Omega)) \leq h\psi(\Omega)
\]
for any set \( \Omega \subset \text{Dom}(f) \). Let \( Q \) be a topological space and let \( \Omega_0 \) be a nonempty subset of \( X \). A continuous map \( K : \Omega_0 \times Q \to X \) is said to be \( \psi \) condensing with respect to the first variable if
\[
\psi(K(\Omega, C)) \leq h\psi(\Omega)
\]
for any compact \( C \subset Q \) and \( \Omega \subset \Omega_0 \).

**Definition 1.6.** Let \( f \) be a continuous operator acting from the closure \( \overline{U} \) of a bounded open subset \( U \) of a Banach space \( X \) into \( X \), \( \psi \) condensing with respect to a monotone MNC \( \psi \) and without fixed points on the boundary \( \partial U \) of \( U \). Then one can define an integer valued characteristic \( \text{ind}(f, U) \) called the index of \( f \) in \( U \), which enjoys all the usual properties of the index (see [1]).

3. Results

**Theorem 1.1.** Let us consider the following boundary value problem
\[
(BV) \begin{cases} 
\dot{x} = f(t, x) & t \in [a, b] = I \subset \mathbb{R}, \ x \in X, \\
x \in S 
\end{cases}
\]
\( X \) is a Banach space, \( f : (t, x) \to f(t, x) \in C(I \times X, X) \) and \( S \subset C(I, X) \). Let us assume that there exists a closed bounded convex set \( Q \subset C(I, X) \) and a closed set \( S_1 \subset S \cap Q \), such that the solutions of the integral equation
\[
(I) \quad x = K(x, q)
\]
are also solution of the following "linearized" boundary value problem
\[
\begin{cases} 
\dot{x} = g(t, x, q) & t \in I, \ x \in X, \\
x \in S_1 
\end{cases}
\]
for any \( q \in Q \); the operator \( K : \Omega \times Q \to C(I, X) \) satisfies the following condition:

(C) \( K \) is condensing in the first variable with respect to a monotone and regular MNC \( \psi \),

and \( \Omega \) is an open bounded and convex subset of \( X \) such that

\[
\text{ind} \ (K(\cdot, q), \Omega) \neq 0
\]

for some (and hence for all) \( q \in Q \), and the function \( g : I \times X^2 \to X \) is continuous and such that

\[
g(t, x, x) = f(t, x)
\]

for any \( t \in I, x \in X \). Let \( \Sigma : Q \to Q \) be the operator which maps each \( q \in Q \) into the set of solutions of \((I)\). Then if we assume that the following condition

(i) for each \( q \in Q \) the set \( \Sigma(q) \) is a set of isolated points

holds, problem \((BV)\) has a solution.

**Proof.** We show at first that \( \Sigma \) is an u.s.c. multivalued map from \( Q \) into \( Q \). Let \( D \) be a closed subset of \( Q \), \( \{q_n\}_{n \in \mathbb{N}} \subset \Sigma^{-1}(D) \) such that \( q_n \to q_0 \). Choose \( x_n \in \Sigma(q_n) \cap D \), that is

\[
x_n = K(x_n, q_n)
\]

for any \( n \in \mathbb{N} \).

It follows that

\[
\bigcup_{n \in \mathbb{N}} x_n \subset K \left( \bigcup_{n \in \mathbb{N}} x_n, \bigcup_{n \in \mathbb{N}} q_n \right),
\]

and as \( K \) is \( \psi \) condensing in the first variable

\[
\psi \left( \bigcup_{n \in \mathbb{N}} x_n \right) \leq h \psi \left( \bigcup_{n \in \mathbb{N}} x_n \right),
\]

that is, \( \bigcup_{n \in \mathbb{N}} x_n \) is relatively compact. Without loss of generality, we can assume that \( x_n \to x_0 \) and then passing to the limit for \( n \to +\infty \) in \((1.2)\), we have \( x_0 \in \Sigma(q_0) \cap D \), i.e., \( q_0 \in \Sigma^{-1}(d) \), so that \( \Sigma \) is u.s.c.. We want to show that \( \Sigma \) is a w-map in the Darbo sense. Fix \( q \in Q \) and choose \( y \in \Sigma(q) \); as by hypothesis \( y \) is an isolated solution, there will exist an open set \( \Omega_y \subset \Omega \) such that \( \Sigma(q) \cap \Omega_y = \{y\} \). We define the integer

\[
n(y, \Sigma(y)) = \text{ind} \ (K_q, \Omega_q),
\]

where \( K_q : \Omega \to C(I, X) \) is defined by \( K_q(x) = K(x, q) \).

By the excision property of the index of condensing operators, \( n(y, \Sigma(y)) \) does not depend on the choice of \( \Omega_1 \subset \Omega \). Let now \( W \subset \Omega \) be an open set such that \( \Sigma(q) \cap \partial W = \emptyset \). As \( \Sigma \) is u.s.c., it there exists \( B(q, r) \) such that for any \( q' \in B(q, r) \cap Q \)
we have $\Sigma(q') \cap \partial W = \emptyset$. Then we can define the following admissible homotopy between $K_{q|W}$ and $K_{q'|W}$, where $q'$ is fixed in $B(q, r) \cap Q$:

$$H(y, t) = K(y, tq + (1 - t)q') \quad y \in W, \; t \in [0, 1].$$

For the additivity property of the index we have

$$\sum_{y \in \Sigma(q) \cap \partial W} n(y, \Sigma(q)) = \text{ind}(K_q, W) = \text{ind}(K_{q'}, W) = \sum_{y \in \Sigma(q') \cap \partial W} n(y, \Sigma(q')).$$

Thus $\Sigma$ is a w-map where $i(\Sigma) = \text{ind}(K_q, \Omega)$, and it is possible (see [7]) to say that $\Sigma$ has the fixed point property, and by (1.1) the theorem is proved.

With a proof similar to that used in Theorem 1.1 it is possible to obtain the following result:

**Proposition 1.2.** Consider the following boundary value problem

$$\begin{cases}
  x'' = f(t, x, x', x''), & t \in I = [a, b], \; x \in X, \\
  x \in S
\end{cases}$$

where $(t, x, x', x'') \rightarrow f(t, x, x', x'') \in C(I \times X^3, X)$ and $S \subset C^1(I, X)$. Assume that there exist a bounded closed and convex subset $Q \subset C^2(I, X)$ and a closed subset $S_1 \subset S \cap Q$ such that the solutions of the following problem

$$\begin{cases}
  x'' = f(t, x, q', q''), & t \in I = [a, b] \subset \mathbb{R}, \; q \in X, \\
  x \in S_1
\end{cases}$$

(include the solutions of some integral equation

$$x = K(x, q', q'$$

for all $q \in Q$, where $K : \Omega \times Q \rightarrow C(I, X)$ satisfies condition (C) and $\text{ind}(K(\cdot, q)) \neq 0$ for some (and hence for all) $q \in Q$, and for some open and convex set $\Omega \subset C(I, X)$.

Let $\Sigma : Q \rightarrow Q$ be the operator which maps each $q \in Q$ into the set of solutions of (I2). If $\Sigma$ satisfies assumption (i), then (BV2) has a solution.

The example that we will present in this paper will be an application of the following result, whose proof can be obtained immediately from the one of Theorem 1.1.

**Proposition 1.3.** Theorem 1.1 and Proposition 1.2 still hold if assumption (C) is replaced by the following weaker hypotheses:

- $(C_1)$ $K : \Omega \times Q \rightarrow C(I, X)$ is condensing in the first variable on the equicontinuous subsets of $\Omega$ with respect to a monotone MNC $\psi_1$, regular on equicontinuous subsets;
(C₂) \( K_q : \Omega \rightarrow C(I, X) \) is \( \psi_2 \) condensing, where \( \psi_2 \) is a monotone MNC; and if we assume that \( \Sigma \) satisfies (i), the further assumption

(e) \( \Sigma(Q) \) is an equicontinuous set.

4. An example

Let us consider the problem

\[
\begin{aligned}
(P) \quad &\begin{cases}
x'' = g(t,x,x',x''), \\
x(0) = x(1) = 0,
\end{cases}
\end{aligned}
\]

where \( t \in I = [0, 1] \), \( x \in X \), a weakly compact generated Banach space (i.e. a Banach space that coincides with the linear envelope of a weakly compact subset), \( g : I \times X^3 \rightarrow X \) is an uniformly continuous map such that the following assumptions are satisfied:

(a₁) there exist two positive constants \( m, n \) with \( 0 < n < 8 \) such that

\[
\| g(t,x_1,x_2,x_3) \| < m \| x_1 \| + n
\]

for any \( x_1, x_2, x_3 \in X, t \in I \);

(a₂) there exists a continuous derivative \( g_{x_1}(t,x_1,x_2,x_3) \) of \( g \) with respect to \( x_1 \);

(a₃) there exist \( \phi, \psi, \eta \in L^1(I, R^+) \) such that

\[
\int_0^1 \phi(t) \, dt < 2
\]

and

\[
\beta(g(t,A,B,C)) \leq \phi(t) \beta(A) + \psi(t) \beta(B) + \eta(t) \beta(C)
\]

for any \( t \in I \) and \( A, B, C \subset X \) bounded.

Let us consider for fixed \( q \in C^2(I, X) \) the problem

\[
(P_q) \quad \begin{cases}
x'' = g(t,x,q',q''), \\
x(0) = x(1),
\end{cases}
\]

and assume that \( P_q \) satisfies the following:

(a₄) \((P_q)\) does not present resonance for any \( q \in C^2(I, X) \).

Then the solutions of \((P_q)\) are given by the integral equation (see [6])

\[
(I_q) \quad x(t) = \int_0^1 G(t,s) g(s,x(s),q'(s),q''(s)) \, ds
\]
where \( G(t,s) : I^2 \to \mathbb{R} \) is the Green function defined by

\[
G(t,s) = \begin{cases} 
(t-1)s, & 0 \leq s \leq t \leq 1, \\
(s-1)t, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

We show at first that the possible solution of (I)_q (i.e. (P_q)) are equibounded in \( C^2(I,X) \) so that we can define the set \( Q \).

By (a_1) it follows immediately that if \( x \) is a solution of (I)_q we have

\[
\|x\| \leq \frac{n}{8-m} = M_0,
\]

and from the differential equation of (P_q), still using (a_1), we get

\[
\|x''\| \leq mM_0 + n = M_1.
\]

Now we fix \( \bar{t} \in I \) and let \( L \in X^* \) such that \( \|L\| = 1 \) and \( L(x'(\bar{t})) = \|x'(\bar{t})\| \). The function \( t \to L(x(t)) \) satisfies the problem

\[
\begin{cases}
L''(x(t)) = L(g(t,x(t),q'(t),q''(t)), \\
L(x(0)) = L(x(1)) = 0.
\end{cases}
\]

Then there will exist \( \xi \in (0,1) \) such that \( L'(x(t))_{|t=\xi} = L(x'(\xi)) \), and we have

\[
L'(x(t)) = L(x'(t)) = \int_{\xi}^{t} L''(x(s)) \, ds = \int_{\xi}^{t} L(x''(s)) \, ds = L\left( \int_{\xi}^{t} x''(s) \, ds \right).
\]

It follows that

\[
\|x'(\bar{t})\| = L(x'(\bar{t})) \leq \|x''\| \leq M_1.
\]

By the arbitrariness of \( \bar{t} \in I \) we obtain \( \|x'\| \leq M_1 \). We let \( M = \max\{M_0, M_1\} \) and we define the set

\[
Q = \{x \in C^2(I,X) : \max\{\|x\|, \|x'\|, \|x''\|\} \leq M\}.
\]

Now we prove that \( \Sigma : Q \to Q \) is such that \( \Sigma(Q) \) is equicontinuous in \( C(I,X) \), so that condition (e) is satisfied. In fact we have

\[
\|x(t_1) - x(t_2)\| = \left\| \int_0^1 [G(t_1,s) - G(t_2,s)] g(s,x(s),q'(s),q''(s)) \, ds \right\|
\]

\[
\leq \int_0^1 |G(t_1,s) - G(t_2,s)| \|g(s,x(s),q'(s),q''(s))\| \, ds < \epsilon (mM + n)
\]

if \( |t_1 - t_2| < \delta_\epsilon \) for a suitable \( \delta_\epsilon > 0 \).

We let

\[
\Omega = B(0,M) = \{x \in C(I,X) : \|x\| < M\},
\]
and let $K : \Omega \times Q \to C(I, X)$ the operator defined by
\[
K(x, q)(t) = \int_0^1 G(t, s) g(s, x(s), q'(s), q''(s)) \, ds.
\]

We will show that $K$ satisfies the conditions (C₁) and (C₂). In the following, if $\Omega \subset C(I, X)$,
\[
\Omega(s) = \{ x(s), \ x \in \Omega \}.
\]

It is easy to see that $K$ is a continuous operator. Let $H \subset \Omega$ be an equicontinuous set and let $C \subset Q$ be a compact set. Then for a "$t$" fixed the set of functions
\[
\{ G(t, s) g(s, x(s), q'(s), q''(s)) : x \in H, \ q \in C \}
\]
is an equicontinuous one, so that it is possible to interchange the $\beta$ MNC with the integral sign, obtaining
\[
\beta \left( \left\{ \int_0^1 G(t, s) g(s, x(s), q'(s), q''(s)) \, ds \ : \ x \in H, \ q \in C \right\} \right)
\leq \int_0^1 |G(t, s)| \beta(\{ g(s, x(s), q'(s), q''(s)) \, ds \ : \ x \in H, \ q \in C \}),
\]
and by (a₂) we have
\[
\beta(\{ K(x, q)(t), \ x \in H, \ q \in C \}) \leq
\leq \int_0^1 |G(t, s)| [\phi(s) \beta(H(s)) + \psi(s) \beta(C'(s)) + \eta(s) \beta(C''(s))] \, ds =
= \int_0^1 |G(t, s)| \phi(s) \beta(H(s)) \, ds, \quad \forall \ t \in I
\]
as $\{ C'(s) \}, \{ C''(s) \}$ are compact sets in $X$ for any $s \in I$.

If we let $\beta_1(\Omega) = \sup_{t \in I} \beta(\Omega(t))$, $\Omega \subset C(I, X)$ be bounded, from the previous inequality we obtain, if we let $h = \frac{1}{2} \int_0^1 \phi(s) \, ds$:
\[
\beta_1(K(H, G)) \leq h \beta_1(H),
\]
that is, as (a₃) holds, we have proved that (C₁) is satisfied.

In order to prove that the operator $K_q : \Omega \to C(I, X)$ defined by $K_q : x \to K(x, q)$, satisfies (C₂), we introduce the following monotone MNC:
\[
\beta_2(H) = \sup \{ \beta_1(H) \ | \ E \text{ is a countable subset of } H \}
\]
where $H \subset C(I, X)$ is bounded. Let $H \subset \Omega$ be bounded. Let $Y$ be a countable subset of $K_q(H)$ and let $Z \subset H$ be such that $Z$ is countable and $K_q(Z) = Y$. 
As $X$ is a weakly compact generated Banach space it follows that (see [9])
\[
\beta\left(\left\{ \int_0^1 G(t, s) g(s, z(s), q'(s), q''(s)) \, ds, \ z \in Z \right\} \right)
\leq \int_0^1 |G(t, s)| \beta\left(\{ g(s, z(s), q'(s), q''(s)), \ z \in Z \right\}) \, ds
\]
so that, again by (a3) we obtain, considering the supremum with respect to $t$ in the inequality
\[
\beta_1(Y) \leq h \beta_1(Z) \leq h \beta_2(H).
\]
As $Y$ was an arbitrary countable subset in $K_q(H)$ we get
\[
\beta_2(K_q(H)) \leq h \beta_2(H)
\]
so that (C2) holds. Then the index $\text{ind}(K_q, \Omega)$ is defined and, considering the admissible homotopy
\[
H(\lambda, x) = \lambda K_q(x) \quad \lambda \in [0, 1], \ x \in \Omega,
\]
we have
\[
\text{ind}(K(\cdot, q), \Omega) = 1.
\]
At last we show that the integral equation has only isolated solutions. In fact the Frechét derivative of $K_q$, calculated in a solution of $(I_q) x_0$, is given by the following
\[
[K'_q(x_0)](h)(t) = \int_0^1 G(t, s) g_{x_1}(s, x_0(s), q'(s), q''(s)) h(s) \, ds,
\]
and the hypothesis of non resonance implies that $I - K'_q(x_0)$ is invertible, that is $x_0$ is isolated. Then, by Proposition 1.3, the problem $(P_q)$ has solution.

REFERENCES


*Manuscript received April 22, 1993*

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