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SOLUTION SETS AND BOUNDARY VALUE PROBLEMS IN BANACH SPACES

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Dedicated to the memory of Juliusz Schauder

1. Introduction

In this paper we present an extension to an arbitrary Banach space X of some results (Theorem 1.2 and Proposition 1.2) contained in [2] with $X = \mathrm{R}^n$, that is we deal with the problem

$$\text{(BV)} \qquad \begin{cases} \dot{x} = f(t,x) & t \in [a,b] = I \subset \mathrm{R}, \quad x \in X, \\ x \in S \end{cases}$$

where X is a Banach space, $f : I \times X \to X$ is a continuous map and S is a subset of the Banach space $C(I,X)$ of continuous functions from I to X with the maximum norm. The extension is obtained in a quite natural way by using condensing operators and the related fixed point theory. We look for solution of (BV) in the form of fixed points of a finite valued upper semicontinuous multivalued map Σ, that is the solution map of a suitable "linearized" problem associated to (BV) (see e.g. [3], [4], [5], [6]). So, in this work, we give an existence result for

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(BV) as well as for the second order boundary value problem

$$\begin{cases} x'' = f(t, x, x', x'') & t \in I \subset \mathbb{R}, \quad x \in X, \\ x \in S & S \subset C^1(I, X), \end{cases}$$

generalizing Proposition 1.2 in [2]. An example will be given in order to see how this latter result can be applied.

2. Definitions

DEFINITION 1.1. *Let X and Y be metric spaces. A set valued map $\Sigma : X \multimap Y$, with nonempty values, is said to be upper semicontinuous at $x \in X$ if for any neighborhood V of $\Sigma(x)$ there exists a neighborhood U of X such that $\Sigma(x) \subset V$ for any $x \in U$. If for every $x \in X$, Σ is upper semicontinuous at x, then Σ is said to be upper semicontinuous* (u.s.c.) *on X. It is well known that Σ is* u.s.c. *if and only if for any closed subset $D \subset Y$ the set $\Sigma_{-}^{-1}(D) = \{x \in X : \Sigma(x) \cap D \neq \emptyset\}$ is closed in X.*

DEFINITION 1.2. *Let X and Y be topological Hausdorff spaces. A finite valued upper semicontinuous map $\Sigma : X \multimap Y$ will be called a weighted map (shortly w-map) if, to each x and $y \in \Sigma(x)$, a multiplicity of weight $m(y, \Sigma(x)) \in \mathbb{Z}$ is assigned in such a way that the following property holds:If U is an open set in Y with $\partial U \cap \Sigma(x) = \emptyset$, then*

$$\sum_{y \in \Sigma(x) \cap U} m(y, \Sigma(x)) = \sum_{y' \in \Sigma(x') \cap U} m(y, \Sigma(x'))$$

whenever x' is close enough to x (see [6], [7]*).*

DEFINITION 1.3. *The number $i(\Sigma(x), U) = \sum_{y \in \Sigma(x) \cap U} m(y, \Sigma(x))$ will be called the "index" of multiplicity of $\Sigma(x)$ in U. If U is a connected set, the number $i(\Sigma(x))$ does not depend on $x \in X$. In this case the number $i(\Sigma) = i(\Sigma(x), U)$ will be called the index of the weighted map Σ.*

DEFINITION 1.4. *Let X be a Banach space, $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\psi : 2^X \to \mathcal{A}$ is said to be a measure of non compactness* (MNC) *if*

$$\psi(\overline{\mathrm{co}\,\Omega}) = \psi(\Omega) \qquad \text{for every } \Omega \in 2^X .$$

A measure of non compactness is called monotone if $\Omega_0, \Omega_1 \in 2^X$ and $\Omega_0 \subset \Omega_1$ imply $\psi(\Omega_0) \leq \psi(\Omega_1)$.

A real valued MNC *$\psi : 2^X \to [0, +\infty)$ is called regular if $\psi(\Omega) = 0$ is equivalent to the relative compactness of Ω.*

Well known examples of MNC monotone and regular are the following (see [1]):

(a) the Kuratowski MNC defined by $\alpha(\Omega) = \inf\{d > 0 : \Omega$ admits a partition into a finite number of sets whose diameters are less than $d\}$;

(b) the Hausdorff MNC defined by $\beta(\Omega) = \inf\{\epsilon > 0 : \Omega$ has finite ϵ net$\}$. In the following ψ will be a real valued MNC.

DEFINITION 1.5. *Let X be a Banach space and let ψ be a* MNC. *A continuous map $f : \mathrm{dom}\,(f) \subset X \to X$ is said to be ψ condensing if there exists $0 \leq h < 1$ such that*

$$\psi(f(\Omega)) \leq h\psi(\Omega)$$

for any set $\Omega \subset \mathrm{Dom}\,(f)$. Let Q be a topological space and let Ω_0 be a nonempty subset of X. A continuous map $K : \Omega_0 \times Q \to X$ is said to be ψ condensing with respect to the first variable if

$$\psi(K(\Omega, C)) \leq h\psi(\Omega)$$

for any compact $C \subset Q$ and $\Omega \subset \Omega_0$.

DEFINITION 1.6. *Let f be a continuous operator acting from the closure $\overline{U}$ of a bounded open subset U of a Banach space X into X, ψ condensing with respect to a monotone* MNC *ψ and without fixed points on the boundary ∂U of U. Then one can define an integer valued characteristic* $\mathrm{ind}\,(f, U)$ *called the index of f in U, which enjoys all the usual properties of the index (see [1]).*

3. Results

THEOREM 1.1. *Let us consider the following boundary value problem*

$$\text{(BV)} \qquad \begin{cases} \dot{x} = f(t, x) & t \in [a, b] = I \subset \mathrm{R},\ x \in X, \\ x \in S \end{cases}$$

X is a Banach space, $f : (t, x) \to f(t, x) \in C(I \times X, X)$ and $S \subset C(I, X)$. Let us assume that there exists a closed bounded convex set $Q \subset C(I, X)$ and a closed set $S_1 \subset S \cap Q$, such that the solutions of the integral equation

$$\text{(I)} \qquad x = K(x, q)$$

are also solution of the following "linearized" boundary value problem

$$\begin{cases} \dot{x} = g(t, x, q) & t \in I,\ x \in X, \\ x \in S_1 \end{cases}$$

for any $q \in Q$; the operator $K : \Omega \times Q \to C(I, X)$ satisfies the following condition:

(C) *K is condensing in the first variable with respect to a monotone and regular MNC ψ,*

and Ω is an open bounded and convex subset of X such that

$$\operatorname{ind}(K(\cdot, q), \Omega) \neq 0 \tag{1.1}$$

for some (and hence for all) $q \in Q$, and the function $g : I \times X^2 \to X$ is continuous and such that

$$g(t, x, x) = f(t, x)$$

for any $t \in I$, $x \in X$. Let $\Sigma : Q \to Q$ be the operator which maps each $q \in Q$ into the set of solutions of (I). *Then if we assume that the following condition*

(i) *for each $q \in Q$ the set $\Sigma(q)$ is a set of isolated points*

holds, problem (BV) *has a solution.*

PROOF. We show at first that Σ is an u.s.c. multivalued map from Q into Q. Let D be a closed subset of Q, $\{q_n\}_{n\in \mathrm{N}} \subset \Sigma_-^{-1}(D)$ such that $q_n \to q_0$. Choose $x_n \in \Sigma(q_n) \cap D$, that is

$$x_n = K(x_n, q_n) \qquad \text{for any } n \in \mathrm{N}. \tag{1.2}$$

It follows that

$$\bigcup_{n\in\mathrm{N}} x_n \subset K\Big(\bigcup_{n\in\mathrm{N}} x_n, \bigcup_{n\in\mathrm{N}} q_n\Big),$$

and as K is ψ condensing in the first variable

$$\psi\Big(\bigcup_{n\in\mathrm{N}} x_n\Big) \leq h\,\psi\Big(\bigcup_{n\in\mathrm{N}} x_n\Big),$$

that is, $\bigcup_{n\in\mathrm{N}} x_n$ is relatively compact. Without loss of generality, we can assume that $x_n \to x_0$ and then passing to the limit for $n \to +\infty$ in (1.2), we have $x_0 \in \Sigma(q_0) \cap D$, i.e., $q_0 \in \Sigma_-^{-1}(d)$, so that Σ is u.s.c.. We want to show that Σ is a w-map in the Darbo sense. Fix $q \in Q$ and choose $y \in \Sigma(q)$; as by hypothesis y is an isolated solution, there will exist an open set $\Omega_q \subset \Omega$ such that $\Sigma(q) \cap \overline{\Omega}_1 = \{y\}$. We define the integer

$$n(y, \Sigma(y)) = \operatorname{ind}(K_q, \Omega_q),$$

where $K_q : \Omega \to C(I, X)$ is defined by $K_q(x) = K(x, q)$.

By the excision property of the index of condensing operators, $n(y, \Sigma(y))$ does not depend on the choice of $\Omega_1 \subset \Omega$. Let now $W \subset \Omega$ be an open set such that $\Sigma(q) \cap \partial W = \emptyset$. As Σ is u.s.c., it there exists $B(q, r)$ such that for any $q' \in B(q, r) \cap Q$

we have $\Sigma(q') \cap \partial W = \emptyset$. Then we can define the following admissible homotopy between $K_{q|W}$ and $K_{q'|W}$, where q' is fixed in $B(q,r) \cap Q$:

$$H(y,t) = K(y, tq + (1-t)q') \qquad y \in W, \ t \in [0,1].$$

For the additivity property of the index we have

$$\sum_{y \in \Sigma(q) \cap W} n(y, \Sigma(q)) = \operatorname{ind}(K_q, W) = \operatorname{ind}(K_{q'}, W) = \sum_{y \in \Sigma(q') \cap W} n(y, \Sigma(q')).$$

Thus Σ is a w-map where $i(\Sigma) = \operatorname{ind}(K_q, \Omega)$, and it is possible (see [7]) to say that Σ has the fixed point property, and by (1.1) the theorem is proved.

With a proof similar to that used in Theorem 1.1 it is possible to obtain the following result:

PROPOSITION 1.2. *Consider the following boundary value problem*

$$\begin{cases} x'' = f(t, x, x', x''), & t \in I = [a,b], \ x \in X, \\ x \in S \end{cases}$$

where $(t, x, x', x'') \to f(t, x, x', x'') \in C(I \times X^3, X)$ *and* $S \subset C^1(I, X)$. *Assume that there exist a bounded closed and convex subset* $Q \subset C^2(I, X)$ *and a closed subset* $S_1 \subset S \cap Q$ *such that the solutions of the following problem*

$$\text{(BV2)} \qquad \begin{cases} x'' = f(t, x, q', q''), & t \in I = [a,b] \subset \mathrm{R}, \ q \in X, \\ x \in S_1 \end{cases}$$

include the solutions of some integral equation

$$\text{(I2)} \qquad x = K(x, q', q'')$$

for all $q \in Q$, *where* $K : \Omega \times Q \to C(I, X)$ *satisfies condition* (C) *and* $\operatorname{ind}(K(\cdot, q)) \neq 0$ *for some (and hence for all)* $q \in Q$, *and for some open and convex set* $\Omega \subset C(I, X)$.

Let $\Sigma : Q \to Q$ *be the operator which maps each* $q \in Q$ *into the set of solutions of* (I2). *If* Σ *satisfies assumption* (i), *then* (BV2) *has a solution.*

The example that we will present in this paper will be an application of the following result, whose proof can be obtained immediately from the one of Theorem 1.1.

PROPOSITION 1.3. *Theorem* 1.1 *and Proposition* 1.2 *still hold if assumption* (C) *is replaced by the following weaker hypotheses:*

- (C_1) $K : \Omega \times Q \to C(I, X)$ *is condensing in the first variable on the equicontinuous subsets of* Ω *with respect to a monotone* MNC ψ_1, *regular on equicontinuous subsets;*

(C$_2$) *$K_q : \Omega \to C(I,X)$ is ψ_2 condensing, where ψ_2 is a monotone* MNC*;*

and if we assume that Σ satisfies (i)*, the further assumption*

(e) *$\Sigma(Q)$ is an equicontinuous set.*

4. An example

Let us consider the problem

$$\text{(P)} \qquad \begin{cases} x'' = g(t, x, x', x''), \\ x(0) = x(1) = 0, \end{cases}$$

where $t \in I = [0,1]$, $x \in X$, a weakly compact generated Banach space (i.e. a Banach space that coincides with the linear envelope of a weakly compact subset), $g : I \times X^3 \to X$ is an uniformly continuous map such that the following assumptions are satisfied:

(a$_1$) there exist two positive constants m, n with $0 < n < 8$ such that

$$\| g(t, x_1, x_2, x_3) \| \leq m \| x_1 \| + n$$

for any $x_1, x_2, x_3 \in X$, $t \in I$;

(a$_2$) there exists a continuous derivative $g_{x_1}(t, x_1, x_2, x_3)$ of g with respect to x_1;

(a$_3$) there exist $\phi, \psi, \eta \in L^1(I, \mathrm{R}^+)$ such that

$$\int_0^1 \phi(t)\, dt < 2$$

and

$$\beta(g(t, A, B, C)) \leq \phi(t)\, \beta(A) + \psi(t)\, \beta(B) + \eta(t)\, \beta(C)$$

for any $t \in I$ and $A, B, C \subset X$ bounded.

Let us consider for fixed $q \in C^2(I, X)$ the problem

$$(\mathrm{P}_q) \qquad \begin{cases} x'' = g(t, x, q', q''), \\ x(0) = x(1), \end{cases}$$

and assume that (P$_q$) satisfies the following:

(a$_4$) (P$_q$) does not present resonance for any $q \in C^2(I, X)$.

Then the solutions of (P$_q$) are given by the integral equation (see [8])

$$(\mathrm{I}_q) \qquad x(t) = \int_0^1 G(t, s)\, g(s, x(s), q'(s), q''(s))\, ds$$

where $G(t,s) : I^2 \to \mathrm{R}$ is the Green function defined by

$$G(t,s) = \begin{cases} (t-1)s, & 0 \le s \le t \le 1, \\ (s-1)t, & 0 \le t \le s \le 1. \end{cases}$$

We show at first that the possible solution of (I_q) (i.e. (P_q)) are equibounded in $C^2(I,X)$ so that we can define the set Q.

By (a_1) it follows immediately that if x is a solution of (I_q) we have

$$\| x \| \le \frac{n}{8-m} = M_0,$$

and from the differential equation of (P_q), still using (a_1), we get

$$\| x'' \| \le mM_0 + n = M_1.$$

Now we fix $\bar{t} \in I$ and let $L \in X^*$ such that $\| L \| = 1$ and $L(x'(\bar{t})) = \| x'(\bar{t}) \|$. The function $t \to L(x(t))$ satisfies the problem

$$\begin{cases} L''(x(t)) = L(g(t,x(t),q'(t),q''(t)), \\ L(x(0)) = L(x(1)) = 0. \end{cases}$$

Then there will exist $\xi \in (0,1)$ such that $L'(x(t))_{|\,t=\xi} = L(x'(\xi))$, and we have

$$L'(x(t)) = L(x'(t)) = \int_\xi^t L''(x(s))\,ds = \int_\xi^t L(x''(s))\,ds = L\left(\int_\xi^t x''(s)\,ds\right).$$

It follows that

$$\| x'(\bar{t}) \| = L(x'(\bar{t})) \le \| x'' \| \le M_1.$$

By the arbitrarity of $\bar{t}$ in I we obtain $\| x' \| \le M_1$. We let $M = \max\{M_0, M_1\}$ and we define the set

$$Q = \{ x \in C^2(I,X) \,:\, \max\{\| x \|, \| x' \|, \| x'' \|\} \le M \}.$$

Now we prove that $\Sigma : Q \multimap Q$ is such that $\Sigma(Q)$ is equicontinuous in $C(I,X)$, so that condition (e) is satisfied. In fact we have

$$\|x(t_1) - x(t_2)\| = \left\| \int_0^1 [G(t_1,s) - G(t_2,s)]\, g(s,x(s),q'(s),q''(s))\,ds \right\|$$

$$\le \int_0^1 |G(t_1,s) - G(t_2,s)|\, \|g(s,x(s),q'(s),q''(s))\|\,ds < \epsilon(mM+n)$$

if $|t_1 - t_2| < \delta_\epsilon$ for a suitable $\delta_\epsilon > 0$.

We let

$$\Omega = B(0,M) = \{ x \in C(I,X) \,:\, \|x\| < M\},$$

and let $K : \Omega \times Q \to C(I, X)$ the operator defined by

$$K(x,q)(t) = \int_0^1 G(t,s)\, g(s, x(s), q'(s), q''(s))\, ds.$$

We will show that K satisfies the conditions (C_1) and (C_2). In the following, if $\Omega \subset C(I, X)$,

$$\Omega(s) = \{x(s),\ x \in \Omega\}.$$

It is easy to see that K is a continuous operator. Let $H \subset \Omega$ be an equicontinuous set and let $C \subset Q$ be a compact set. Then for a "t" fixed the set of functions

$$\{G(t,s)\, g(s, x(s), q'(s), q''(s)) \ :\ x \in H,\ q \in C\}$$

is an equicontinuous one, so that it is possible to interchange the β MNC with the integral sign, obtaining

$$\beta\Big(\Big\{ \int_0^1 G(t,s)\, g(s, x(s), q'(s), q''(s))\, ds \ \ x \in H,\ q \in C\Big\}\Big)$$
$$\leq \int_0^1 |G(t,s)|\, \beta(\{g(s, x(s), q'(s), q''(s))\, ds \ \ x \in H,\ q \in C\}),$$

and by (a_2) we have

$$\beta(\{K(x,q)(t),\ x \in H,\ q \in C\}) \leq$$
$$\leq \int_0^1 |G(t,s)|\, [\phi(s)\beta(H(s)) + \psi(s)\beta(C'(s)) + \eta(s)\beta(C''(s))]\, ds =$$
$$= \int_0^1 |G(t,s)|\, \phi(s)\, \beta(H(s))\, ds, \qquad \forall\, t \in I$$

as $\{C'(s)\}$, $\{C''(s)\}$ are compact sets in X for any $s \in I$.

If we let $\beta_1(\Omega) = \sup_{t\in I} \beta(\Omega(t))$, $\Omega \subset C(I, X)$ be bounded, from the previous inequality we obtain, if we let $h = \frac{1}{2}\int_0^1 \phi(s)\, ds$:

$$\beta_1(K(H, G)) \leq h\, \beta_1(H),$$

that is, as (a_3) holds, we have proved that (C_1) is satisfied.

In order to prove that the operator $K_q : \Omega \to C(I, X)$ defined by $K_q : x \to K(x, q)$, satisfies (C_2), we introduce the following monotone MNC:

$$\beta_2(H) = \sup\{\beta_1(H) \ \mid\ E \text{ is a countable subset of } H\}$$

where $H \subset C(I, X)$ is bounded. Let $H \subset \Omega$ be bounded. Let Y be a countable subset of $K_q(H)$ and let $Z \subset H$ be such that Z is countable and $K_q(Z) = Y$.

As X is a weakly compact generated Banach space it follows that (see [9])

$$\beta\Big(\Big\{\int_0^1 G(t,s)\,g(s,z(s),q'(s),q''(s))\,ds,\ z\in Z\Big\}\Big)$$
$$\leq \int_0^1 |G(t,s)|\,\beta(\{\,g(s,z(s),q'(s),q''(s)),\ z\in Z\,\})\,ds$$

so that, again by (a_3) we obtain, considering the supremum with respect to t in the inequality

$$\beta_1(Y) \leq h\,\beta_1(Z) \leq h\,\beta_2(H).$$

As Y was an arbitrary countable subset in $K_q(H)$ we get

$$\beta_2(K_q(H)) \leq h\,\beta_2(H)$$

so that (C_2) holds. Then the index $\mathrm{ind}\,(K_q,\Omega)$ is defined and, considering the admissible homotopy

$$H(\lambda,x) = \lambda\,K_q(x) \qquad \lambda\in[0,1],\ x\in\Omega,$$

we have

$$\mathrm{ind}\,(K(\cdot,q),\Omega) = 1.$$

At last we show that the integral equation has only isolated solutions. In fact the Frechet derivative of K_q, calculated in a solution of (I_q) x_0, is given by the following

$$[K_q'(x_0)](h)(t) = \int_0^1 G(t,s)\,g_{x_1}(s,x_0(s),q'(s),q''(s))h(s)\,ds,$$

and the hypothesis of non resonance implies that $I - K_q'(x_0)$ is invertible, that is x_0 is isolated. Then, by Proposition 1.3, the problem (P_q) has solution.

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