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FUNCTORALITY AND HEAT CONTENT ASYMPTOTICS FOR OPERATORS OF LAPLACE TYPE

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(*Submitted by A. Granas*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

Let M be a compact Riemannian manifold of dimension m with C^∞ boundary ∂M. Let V be a smooth vector bundle over M, let ∇^D be a connection on V, and let $E^D \in C^\infty(\mathrm{End}(V))$. Let

$$D = -(\Sigma_{\nu,\mu} g^{\nu\mu} \nabla^D_\nu \nabla^D_\mu + E^D) \tag{1.1}$$

be an operator of Laplace type on $C^\infty(V)$. Note that for every operator D of Laplace type there is a unique connection ∇^D and a unique endomorphism E^D so that (1.1) holds. For example, the geometers Laplacian $D_0 = \delta d$ on $C^\infty(M)$ is an operator of Laplace type with ∇^D trivial and $E^D = 0$. Let $S \in C^\infty(\mathrm{End}(V))$ and let e_m be the inward pointing unit normal. Let

$$\mathcal{B}^+_S f = (\nabla^D_{e_m} f + Sf)|_{\partial M} \quad \text{and} \quad \mathcal{B}^- f = f|_{\partial M} \tag{1.2}$$

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be Neumann and Dirichlet boundary conditions. Let $h(x,t) = e^{-tD_\mathcal{B}} f_1(x)$ for $\mathcal{B} \in \{\mathcal{B}_S^+, \mathcal{B}^-\}$ be the solution for $t > 0$ to the equations:

$$\partial_t h(x,t) = -Dh(x,t), \ \mathcal{B}h(x,t) = 0, \quad \text{and} \quad \lim_{t\to 0^+} h(x,t) = f_1(x). \tag{1.3}$$

Denote the natural pairing between V and the dual bundle V^* by $\langle \cdot, \cdot \rangle$. Let dx and dy be the Riemannian measures on M and ∂M. To study the short time behavior of h we introduce an auxiliary smooth test function $f_2 \in C^\infty(V^*)$ and define

$$\beta(f_1, f_2, D, \mathcal{B})(t) = \int_M \langle e^{-tD_\mathcal{B}} f_1, f_2 \rangle \, dx. \tag{1.4}$$

For example, let $D_0 = \delta d$ be the geometers Laplacian, let $f_1 \in C^\infty(M)$ represent the initial temperature of M, and let ∂M have temperature 0 for all $t > 0$. Then $\beta(f_1, 1, D, \mathcal{B}^-)$ is the total heat energy content of the manifold. We can also study the corresponding localized problem by choosing f_2 suitably.

Standard elliptic methods, see for example the discussion in [4, Lemma 1.3], show there is an asymptotic series as $t \to 0^+$ of the form

$$\beta(f_1, f_2, D, \mathcal{B})(t) \simeq \sum_{n=0}^{\infty} \beta_n(f_1, f_2, D, \mathcal{B}) t^{n/2}, \tag{1.5}$$

and the existence of local invariants $\beta_n^{int}(f_1, f_2, D)$ and $\beta_n^{bd}(f_1, f_2, D, \mathcal{B})$ so that

$$\beta_n(f_1, f_2, D, \mathcal{B}) = \int_M \beta_n^{int}(f_1, f_2, D) \, dx + \int_{\partial M} \beta_n^{bd}(f_1, f_2, D, \mathcal{B}) \, dy. \tag{1.6}$$

We set $t = 0$ to see

$$\beta_0(f_1, f_2, D, \mathcal{B}) = \int_M \langle f_1, f_2 \rangle \, dx. \tag{1.7}$$

We introduce the following notational conventions. Let $x = (x^1, ..., x^m)$ be a system of local coordinates on M. Near ∂M, we choose coordinates $x = (y, x^m)$ so x^m is the geodesic distance to the boundary and so the curves $y(s) = (y, s)$ are unit speed geodesics perpendicular to ∂M. Greek indices ν, μ (resp. Roman indices i, j) range from 1 through m and index coordinate (resp. orthonormal) frames for the tangent $T(M)$ and cotangent $T^*(M)$ bundles of M. Greek indices α, β (resp. Roman indices a, b) range from 1 through $m-1$ and index coordinate (resp. orthonormal) frames for $T(\partial M)$ and $T^*(\partial M)$. We adopt the Einstein convention and sum over repeated indices. If $M = [r_0, r_1] \times M_2$, let $\varepsilon\partial_r$ be the inward unit normal; $\varepsilon(r_0) = 1$ and $\varepsilon(r_1) = -1$.

Let R_{ijkl} be the curvature tensor of the Levi-Civita connection ∇^M of M; with our sign convention $R_{1212} = -1$ for the standard sphere in $\mathbf{R}^3$. Let $\tau = R_{ijji}$ and $\rho_{ij} = R_{ikkj}$ be the scalar curvature and the Ricci tensor. Let ";" and ":" be multiple covariant differentiation with respect to ∇^M and $\nabla^{\partial M}$. When sections of bundles built from V are involved, ";" and ":" will mean

$$\nabla^M \otimes 1 + 1 \otimes \nabla^D \quad \text{and} \quad \nabla^{\partial M} \otimes 1 + 1 \otimes \nabla^D. \tag{1.8}$$

We use the dual connection on V^*. The second fundamental form L measures the difference between ∇^M and $\nabla^{\partial M}$. If $f \in C^\infty(V)$, $f_{;a} = f_{:a}$ since no tangential indices are to be differentiated. However, since the index a must be covariantly differentiated, $f_{;ab} = f_{:ab} - L_{ab} f_{;m}$ where $L_{ab} = (e_m, \nabla^M_{e_a} e_b)$. Let $\widetilde{D}$ be the formal adjoint on $C^\infty(V^*)$; $\nabla^{\widetilde{D}} = \{\nabla^D\}^*$ and $E^{\widetilde{D}} = (E^D)^*$. Let

$$\widetilde{\mathcal{B}}^+_S f_2 = (f_{2;m} + S^* f_2)|_{\partial M} \quad \text{and} \quad \widetilde{\mathcal{B}}^- f_2 = f|_{\partial M} \quad \text{for} \quad f_2 \in C^\infty(V^*). \tag{1.9}$$

The boundary conditions $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are adjoint, i.e. if $\mathcal{B} f_1 = 0$ and $\widetilde{\mathcal{B}} f_2 = 0$, then

$$\int_M \langle D f_1, f_2 \rangle \, dx = \int_M \langle f_1, \widetilde{D} f_2 \rangle \, dx. \tag{1.10}$$

The main results of this paper are the formulae for the invariants

$$\beta_4(f_1, f_2, D, \mathcal{B}^-) \quad \text{and} \quad \beta_6(f_1, f_2, D, \mathcal{B}^+_S) \tag{1.11}$$

in the following theorems.

THEOREM 1.1. (a) $\beta_0(f_1, f_2, D, \mathcal{B}^-) = \int_M \langle f_1, f_2 \rangle \, dx$.

(b) $\beta_1(f_1, f_2, D, \mathcal{B}^-) = -2\pi^{-1/2} \int_{\partial M} \langle f_1, f_2 \rangle \, dy$.

(c) $\beta_2(f_1, f_2, D, \mathcal{B}^-) = -\int_M \langle D f_1, f_2 \rangle \, dx + \int_{\partial M} \{\langle \frac{1}{2} L_{aa} f_1, f_2 \rangle - \langle f_1, f_{2;m} \rangle\} \, dy$.

(d) $\beta_3(f_1, f_2, D, \mathcal{B}^-) = -2\pi^{-1/2} \int_{\partial M} \{\frac{2}{3} \langle f_{1;mm}, f_2 \rangle + \frac{2}{3} \langle f_1, f_{2;mm} \rangle - \langle f_{1:a}, f_{2:a} \rangle$
$+ \langle E^D f_1, f_2 \rangle - \frac{2}{3} L_{aa} \langle f_{1;m}, f_2 \rangle - \frac{2}{3} L_{aa} \langle f_1, f_{2;m} \rangle + \langle (\frac{1}{12} L_{aa} L_{bb} - \frac{1}{6} L_{ab} L_{ab}$
$+ \frac{1}{6} R_{amam}) f_1, f_2 \rangle\} \, dy$.

(e) $\beta_4(f_1, f_2, D, \mathcal{B}^-) = \frac{1}{2} \int_M \langle D f_1, \widetilde{D} f_2 \rangle \, dx + \int_{\partial M} \{\frac{1}{2} \langle (D f_1)_{;m}, f_2 \rangle$
$+ \frac{1}{2} \langle f_1, (\widetilde{D} f_2)_{;m} \rangle - \frac{1}{4} \langle L_{aa} D f_1, f_2 \rangle - \frac{1}{4} \langle L_{aa} f_1, \widetilde{D} f_2 \rangle + \langle (\frac{1}{8} E_{;m} - \frac{1}{16} L_{ab} L_{ab} L_{cc}$
$+ \frac{1}{8} L_{ab} L_{ac} L_{bc} - \frac{1}{16} R_{ambm} L_{ab} + \frac{1}{16} R_{abcb} L_{ac} + \frac{1}{32} \tau_{;m} + \frac{1}{16} L_{ab:ab}) f_1, f_2 \rangle$
$- \frac{1}{4} L_{ab} \langle f_{1:a}, f_{2:b} \rangle - \frac{1}{8} \langle \Omega^D_{am} f_{1:a}, f_2 \rangle + \frac{1}{8} \langle \Omega^D_{am} f_1, f_{2:a} \rangle\} \, dy$.

THEOREM 1.2. (a) $\beta_0(f_1, f_2, D, \mathcal{B}^+_S) = \int_M \langle f_1, f_2 \rangle \, dx$.

(b) $\beta_1(f_1, f_2, D, \mathcal{B}^+_S) = 0$.

(c) $\beta_2(f_1, f_2, D, \mathcal{B}^+_S) = -\int_M \langle D f_1, f_2 \rangle \, dx + \int_{\partial M} \langle \mathcal{B}^+_S f_1, f_2 \rangle \, dy$.

(d) $\beta_3(f_1, f_2, D, \mathcal{B}_S^+) = \frac{2}{3} \cdot 2\pi^{-1/2} \int_{\partial M} \langle \mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle \, dy$.

(e) $\beta_4(f_1, f_2, D, \mathcal{B}_S^+) = \frac{1}{2} \int_M \langle D f_1, \widetilde{D} f_2 \rangle \, dx + \int_{\partial M} \{ -\frac{1}{2} \langle \mathcal{B}_S^+ f_1, \widetilde{D} f_2 \rangle$
$-\frac{1}{2} \langle D f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle + \langle (\frac{1}{2} S + \frac{1}{4} L_{aa}) \mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle \, dy$.

(f) $\beta_5(f_1, f_2, D, \mathcal{B}_S^+) = 2\pi^{-1/2} \int_{\partial M} \{ -\frac{4}{15} \langle \mathcal{B}_S^+ D f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle - \frac{4}{15} \langle \mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+ \widetilde{D} f_2 \rangle$
$-\frac{2}{15} \langle (\mathcal{B}_S^+ f_1)_{:a}, (\widetilde{\mathcal{B}}_S^+ f_2)_{:a} \rangle + \langle (\frac{2}{15} E^D + \frac{4}{15} S^2 + \frac{4}{15} S L_{aa} + \frac{1}{30} L_{aa} L_{bb} + \frac{1}{15} L_{ab} L_{ab}$
$-\frac{1}{15} R_{amam}) \mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle \} \, dy$.

(g) $\beta_6(f_1, f_2, D, \mathcal{B}_S^+) = -\frac{1}{6} \int_M \langle D^2 f_1, \widetilde{D} f_2 \rangle \, dx + \int_{\partial M} \{ \frac{1}{6} \langle \mathcal{B}_S^+ D f_1, \widetilde{D} f_2 \rangle$
$+\frac{1}{6} \langle D^2 f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle + \frac{1}{6} \langle \mathcal{B}_S^+ f_1, \widetilde{D}^2 f_2 \rangle - \frac{1}{6} \langle S \mathcal{B}_S^+ D f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle - \frac{1}{6} \langle S \mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+ \widetilde{D} f_2 \rangle$
$-\frac{1}{12} \langle L_{aa} \mathcal{B}_S^+ D f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle - \frac{1}{12} \langle L_{aa} \mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+ \widetilde{D} f_2 \rangle + \langle (\frac{1}{24} E^D_{;m} + \frac{1}{12} E^D L_{aa}$
$+\frac{1}{48} L_{ab} L_{ab} L_{cc} + \frac{1}{24} L_{ab} L_{ac} L_{bc} - \frac{1}{48} R_{ambm} L_{ab} + \frac{1}{48} R_{abcb} L_{ac} - \frac{1}{24} R_{amam} L_{bb}$
$+\frac{1}{96} \tau_{;m} + \frac{1}{48} L_{ab:ab} + \frac{1}{12} S L_{aa} L_{bb} + \frac{1}{12} S L_{ab} L_{ab} - \frac{1}{12} S R_{amam}$
$+\frac{1}{12} (S E^D + E^D S) + \frac{1}{4} S^2 L_{aa} + \frac{1}{6} S^3 + \frac{1}{6} S_{:aa}) \mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+ f_2 \rangle$
$-\frac{1}{12} L_{aa} \langle (\mathcal{B}_S^+ f_1)_{:b}, (\widetilde{\mathcal{B}}_S^+ f_2)_{:b} \rangle - \frac{1}{12} L_{ab} \langle (\mathcal{B}_S^+ f_1)_{:a}, (\widetilde{\mathcal{B}}_S^+ f_2)_{:b} \rangle$
$-\frac{1}{6} \langle S (\mathcal{B}_S^+ f_1)_{:a}, (\widetilde{\mathcal{B}}_S^+ f_2)_{:a} \rangle - \frac{1}{24} \langle \Omega^D_{am} (\mathcal{B}_S^+ f_1)_{:a}, \widetilde{\mathcal{B}}_S^+ f_2 \rangle$
$+\frac{1}{24} \langle \Omega^D_{am} \mathcal{B}_S^+ f_1, (\widetilde{\mathcal{B}}_S^+ f_2)_{:a} \rangle \} \, dy$.

We refer to [4, Theorem 3.1] for the proof of Theorem 1.1 (a)–(d) and to [7, Theorem 1.2] for the proof of Theorem 1.2 (a)–(f), while Theorem 1.1 (e) was proved in [4, Theorem 1.2 and Lemma 4.3] in the case $f_1 = 1$ and $f_2 = 1$ only. The remainder of this paper is devoted to the proof of Theorem 1.1 (e) and Theorem 1.2 (g). In Section 2, we review the functorial properties of the β_n. In Section 3, we complete the proof. In Section 4, we append some useful combinatorial formulas.

The study of heat content asymptotics is relatively recent. Let

$$\beta(t) = \beta(1, 1, D_0, \mathcal{B}^-)(t) \simeq \sum_{n=0}^{\infty} \beta_n(M) t^{n/2} \tag{1.12}$$

give the asymptotic expansion of the total heat content of M with an initial temperature one. $\beta_0(M)$ and $\beta_1(M)$ for domains in Euclidean space were first computed by van den Berg and Davies [2]; $\beta_2(M)$ was computed for domains in Euclidean space by van den Berg and Le Gall [3]; $\beta_0(M)$, $\beta_1(M)$, and $\beta_2(M)$ were computed by van den Berg [1] for the upper hemisphere of a sphere; these results follow from Theorem 1.1. The case of polygonal domains in the plane was considered by van den Berg and Srisatkunarajah [5]; the analysis is quite different if the boundary is not smooth and these results do not follow from Theorem 1.1. We also refer to related work by Phillips and Jansons [10]. It is possible to generalize these results to mixed boundary conditions, we refer to [7] and to McAvity [9] for details.

The invariants $a_n(M)$ of the heat equation are spectral quantities which are defined by a similar asymptotic series:

$$\operatorname{Tr}_{L^2}(e^{-tD_0}) \simeq (4\pi t)^{-m/2} \sum_{n=0}^{\infty} a_n(M) t^{n/2}. \tag{1.13}$$

The $a_n(M)$ have been used by many authors to prove results in spectral geometry; they also are given by suitable local formulas; see for example [6]. Since computing the β_n seems to be combinatorially more tractible than computing the a_n, it is an interesting open problem to find a simple relationship between them.

2. Functorial properties

We showed [7, Lemma 2.5] the interior integrands have the form:

$$\beta_n^{int}(f_1, f_2, D) = \begin{cases} 0 & \text{if } n = 2k+1, \\ \langle D^k f_1, \widetilde{D}^k f_2\rangle/(2k)! & \text{if } n = 4k, \\ -\langle D^{k+1} f_1, \widetilde{D}^k f_2\rangle/(2k+1)! & \text{if } n = 4k+2. \end{cases} \tag{2.1}$$

Consequently we concentrate on the boundary integrands. The local formula defining $\beta_n^{bd}(f_1, f_2, D, \mathcal{B})$ is built universally and polynomially from the metric tensor, its inverse, and the covariant derivatives of the f_i, the curvature tensor R, the endomorphism E^D, the curvature of the connection ∇^D, the second fundamental form L, and the auxiliary endomorphism S. We set $S = 0$ if $\mathcal{B} = \mathcal{B}^-$; we only tangentially differentiate L and S as these two tensors are only defined on the boundary. By Weyl's work, these polynomials can be formed using only tensor products and contraction of tensor arguments (indices); this yields a Weyl basis. The structure group is the orthogonal group $O(m-1)$ and the normal direction plays a distinguished role. If A is a monomial term of β_n^{bd} of degree $(k_R, k_\Omega, k_E, k_L, k_S)$ in these variables and if k_∇ explicit covariant derivatives appear,

$$2(k_R + k_\Omega + k_E) + k_L + k_S + k_\nabla = n - 1. \tag{2.2}$$

An important observation is that the coefficients which appear relative to a Weyl basis are universal constants independent of the dimension and of the operator D; they depend only on whether Dirichlet or Neumann boundary conditions have been selected. We refer to [7, Lemmas 2.5 and 2.6] for details.

These invariants have many functorial properties.

LEMMA 2.1. (a) $\beta_n(f_1, f_2, D, \mathcal{B}) = \beta_n(f_2, f_1, \widetilde{D}, \widetilde{\mathcal{B}})$.

(b) $\frac{\partial}{\partial\varepsilon}|_{\varepsilon=0}\beta_n(f_1, f_2, D - \varepsilon I, \mathcal{B}) = \beta_{n-2}(f_1, f_2, D, \mathcal{B})$.

(c) *If* $\mathcal{B}f_1 = 0$, *then* $\beta_n(f_1, f_2, D, \mathcal{B}) = -\frac{2}{n}\beta_{n-2}(Df_1, f_2, D, \mathcal{B})$.

PROOF. We refer to [7, §2] for the general case. For ease of exposition, we suppose D is self-adjoint with respect to some fiber metric on V and use the fiber metric to identify V with V^*. Let $\{\phi_\nu, \lambda_\nu\}$ be a spectral resolution of D. The $\{\phi_\nu\}$ are a complete orthonormal basis for $L^2(V)$ of smooth eigensections; $D\phi_\nu = \lambda_\nu\phi_\nu$ and $\mathcal{B}\phi_\nu = 0$. Let $\gamma_\nu^D(f) = \int_M \langle f, \phi_\nu\rangle\, dx$ be the Fourier coefficients. Then

$$e^{-tD_{\mathcal{B}}} f_1 = \Sigma_\nu \gamma_\nu^D(f)\phi_\nu e^{-t\lambda_\nu}, \tag{2.5}$$

$$\beta(f_1, f_2, D, \mathcal{B})(t) = \Sigma_\nu \gamma_\nu^D(f_1)\gamma_\nu^D(f_2)e^{-t\lambda_\nu}; \tag{2.6}$$

(a) follows since (2.0) is symmetric in f_1 and f_2. We replace λ_ν by $\lambda_\nu - \varepsilon$ to see

$$\beta(f_1, f_2, D - \varepsilon I, \mathcal{B})(t) = e^{t\varepsilon}\beta(f_1, f_2, D, \mathcal{B})(t); \tag{2.7}$$

(b) now follows. Finally, if $\mathcal{B}f_1 = 0$, we may use (1.0) to see

$$\gamma_\nu^D(Df_1) = \int_M \langle Df_1, \phi_\nu\rangle\, dx = \int_M \langle f_1, \widetilde{D}\phi_\nu\rangle\, dx = \lambda_\nu\gamma_\nu^D(f_1). \tag{2.8}$$

This implies that:

$$\begin{aligned} -\partial_t\beta(f_1, f_2, D, \mathcal{B})(t) &= \Sigma_\nu \lambda_\nu\gamma_\nu^D(f_1)\gamma_\nu^D(f_2)e^{-t\lambda_\nu} \\ &= \beta(Df_1, f_2, D, \mathcal{B})(t). \end{aligned} \tag{2.9}$$

□

We can link Neumann and Dirichlet boundary conditions. Let $M = [r_0, r_1]$ and let $b \in C^\infty(M)$. Let

$$A = \partial_r + b \text{ and } A^* = \partial_r - b. \tag{2.10}$$

Let $D_1 = A^*A$ and $D_2 = AA^*$ be the associated operators of Laplace type. Let $\varepsilon\partial_r$ be the inward unit normal. Let $S = \varepsilon b$ so $\mathcal{B}_S^+ f = \varepsilon Af|_{\partial M}$.

LEMMA 2.2. $\beta_n(f_1, f_2, D_1, \mathcal{B}_S^+) = -\frac{2}{n}\beta_{n-2}(Af_1, Af_2, D_2, \mathcal{B}^-)$.

PROOF. We compute:

$$-\partial_t\beta(f_1, f_2, D_1, \mathcal{B}_S^+)(t) = \Sigma_\nu \lambda_\nu\gamma_\nu^{D_1}(f_1)\gamma_\nu^{D_1}(f_2)e^{-t\lambda_\nu}. \tag{2.11}$$

We restrict to $\lambda_\nu > 0$ henceforth; the zero spectrum of these operators plays no role. Let $\psi_\nu = \lambda_\nu^{-1/2} A\phi_\nu$; the $\{\psi_\nu, \lambda_\nu\}$ are a spectral resolution for D_2 on range$(A) = \ker(D_2)^\perp$. Since $A\phi_\nu|_{\partial M} = 0$,

$$(2.12)\qquad \begin{aligned}\gamma_\nu^{D_2}(Af) &= \int_M \langle Af, \psi_\nu\rangle\, dx = \lambda_\nu^{-1/2} \int_M \langle Af, A\phi_\nu\rangle\, dx \\ &= \lambda_\nu^{-1/2} \int_M \langle f, A^* A\phi_\nu\rangle\, dx = \lambda_\nu^{1/2}\gamma_\nu^{D_1}(f).\end{aligned}$$

Consequently

$$(2.13)\qquad \begin{aligned}-\partial_t \beta(f_1, f_2, D_1, \mathcal{B}_S^+) &= \Sigma_\nu \gamma_\nu^{D_2}(Af_1)\gamma_\nu^{D_2}(Af_2)e^{-t\lambda_\nu} \\ &= \beta(Af_1, Af_2, D_2, \mathcal{B}^-)(t).\end{aligned}$$

□

REMARK. There are higher dimensional generalizations of this relation involving mixed boundary conditions; we omit details as we shall not need them.

Let $M = M_1 \times M_2$ where the boundary of M_2 is empty. Let $m_i = \dim(M_i)$, let ds_i^2 be the metrics on M_i, and let $V = V_1 \otimes V_2$ where the V_i are smooth bundles over M_i. Let $\sigma, \widehat{\sigma} \in C^\infty(M_1)$ and let $\rho \in C^\infty(M_2)$. Let D_i be operators of Laplace type on V_i, let $S \in C^\infty(V_1)$, let $\mathcal{B} \in \{\mathcal{B}_S^+, \mathcal{B}^-\}$, and let $f_i = \phi_i(x_1) \otimes \psi_i(x_2)$. We shall need several product formulas. The first involves isometric products. Let

$$(2.14)\qquad ds_M^2 = ds_1^2 + ds_2^2 \quad\text{and}\quad D = D_1 \otimes 1 + 1 \otimes D_2.$$

LEMMA 2.3. $\beta_n(f_1, f_2, D, \mathcal{B}) = \Sigma_{p+q=n}\beta_p(\phi_1, \phi_2, D_1, \mathcal{B})\beta_q(\psi_1, \psi_2, D_2)$.

The next lemma involves warped products. Let H be a first order operator on $C^\infty(V_2)$ over M_2. Assume $D_2\psi_1 = H\psi_1 = 0$. Let

$$(2.15)\qquad ds_M^2 = dr^2 + e^{2\sigma} ds_2^2 \text{ and } D = D_1 \otimes 1 + e^{-2\sigma}(1 \otimes (D_2 + \widehat{\sigma} H)).$$

LEMMA 2.4. $\beta_\nu(f_1, f_2, D, \mathcal{B}) = \beta_\nu(\phi_1, e^{m_2\sigma}\phi_2, D_1, \mathcal{B}) \cdot \int_{M_2} \langle \psi_1, \psi_2\rangle\, dx_2$.

Finally, let δ be a small parameter. Assume $D_2\psi_1 = 0$. Let

$$(2.16)\qquad ds_M^2 = ds_1^2 + (1 + \delta\sigma\rho)^2 ds_2^2 \text{ and } D = D_1 \otimes 1 + (1 + \delta\sigma\rho)^{-2} 1 \otimes D_2.$$

LEMMA 2.5. $\beta_\nu(f_1, f_2, D, \mathcal{B}) = \Sigma_k \binom{m_2}{k} \delta^k \beta_\nu(\phi_1, \sigma^k\phi_2, D_1, \mathcal{B}) \int_{M_2} \langle \psi_1, \rho^k \psi_2\rangle\, dx_2$.

PROOFS. To prove Lemma 2.3, we note $e^{-tD_{\mathcal{B}}} = e^{-tD_{1,\mathcal{B}}} \otimes e^{-tD_2}$ so that:

$$(2.17)\qquad \beta(f_1, f_2, D, \mathcal{B})(t) = \beta(\phi_1, \phi_2, D_1, \mathcal{B})(t) \cdot \beta(\psi_1, \psi_2, D_2)(t).$$

Next we prove Lemma 2.4: Since $D_2\psi_1 = 0$,

$$D(\phi \otimes \psi_1) = D_1\phi \otimes \psi_1 \tag{2.18}$$

for $\phi = \phi(x_1)$. Consequently $e^{-tD_{\mathcal{B}}}\phi_1 = e^{-tD_{1,\mathcal{B}}}\phi_1$. Since $dx = e^{n\sigma}\,dr\,dx_2$,

$$\begin{aligned}\beta(f_1, f_2, D, \mathcal{B})(t) &= \int_M \langle e^{-tD_{1,\mathcal{B}}}\phi_1, \phi_2\rangle\langle\psi_1, \psi_2\rangle e^{n\sigma}\,dr\,dx_2\\ &= \beta(\phi_1, e^{n\sigma}\phi_2, D_1, \mathcal{B})(t)\int_{M_2}\langle\psi_1, \psi_2\rangle\,dx_2.\end{aligned} \tag{2.19}$$

Finally, we prove Lemma 2.6. We expand

$$dx = \Sigma_k \tbinom{m_2}{k}\delta^k\sigma^k\rho^k\,dx_1\,dx_2. \tag{2.20}$$

From this we conclude

$$\beta(f_1, f_2, D, \mathcal{B})(t) = \Sigma_k\tbinom{m_2}{k}\beta(\phi_1, \sigma^k\phi_2, D_1, \mathcal{B})\int_{M_2}\langle\psi_1, \rho^k\psi_2\rangle\,dx_2. \tag{2.21}$$

□

3. The proof of Theorems 1.1 and 1.2

We begin with the following technical lemma:

LEMMA 3.1. (a) $\beta_4(f_1, f_2, D, \mathcal{B}^-) = \frac{1}{2}\int_M\langle Df_1, \widetilde{D}f_2\rangle\,dx + \int_{\partial M}\{\frac{1}{2}\langle(Df_1)_{;m}, f_2\rangle$
$+\frac{1}{2}\langle f_1, (\widetilde{D}f_2)_{;m}\rangle - \frac{1}{4}\langle L_{aa}Df_1, f_2\rangle - \frac{1}{4}\langle L_{aa}f_1, \widetilde{D}f_2\rangle + \langle(c_1^- E^D_{;m} + c_2^- E^D L_{aa}$
$+c_3^-\tau L_{aa} + c_4^- L_{aa}L_{bb}L_{cc} + c_5^- L_{ab}L_{ab}L_{cc} + c_6^- L_{ab}L_{ac}L_{bc} + c_7^- R_{ambm}L_{ab}$
$+c_8^- R_{abcb}L_{ac} + c_9^- R_{amam}L_{bb} + c_{10}^-\tau_{;m} + c_{11}^- L_{aa:bb} + c_{12}^- L_{ab:ab})f_1, f_2\rangle$
$+d_1^-\langle L_{aa}f_{1:b}, f_{2:b}\rangle + d_2^- L_{ab}\langle f_{1:a}, f_{2:b}\rangle + d_3^-(\langle\Omega^D_{am}f_{1:a}, f_2\rangle - \langle\Omega^D_{am}f_1, f_{2:a}\rangle)\}\,dy.$

(b) $\beta_6(f_1, f_2, D, \mathcal{B}_S^+) = -\frac{1}{6}\int_M\langle D^2 f_1, \widetilde{D}f_2\rangle\,dx + \int_{\partial M}\{\frac{1}{6}\langle\mathcal{B}_S^+ Df_1, \widetilde{D}f_2\rangle$
$+\frac{1}{6}\langle D^2 f_1, \widetilde{\mathcal{B}}_S^+ f_2\rangle + \frac{1}{6}\langle\mathcal{B}_S^+ f_1, \widetilde{D}^2 f_2\rangle - \frac{1}{6}\langle S\mathcal{B}_S^+ Df_1, \widetilde{\mathcal{B}}_S^+ f_2\rangle$
$-\frac{1}{6}\langle S\mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+\widetilde{D}f_2\rangle - \frac{1}{12}\langle L_{aa}\mathcal{B}_S^+ Df_1, \widetilde{\mathcal{B}}_S^+ f_2\rangle - \frac{1}{12}\langle L_{aa}\mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+\widetilde{D}f_2\rangle$
$+\langle(c_1^+ E^D_{;m} + c_2^+ E^D L_{aa} + c_3^+\tau L_{aa} + c_4^+ L_{aa}L_{bb}L_{cc} + c_5^+ L_{ab}L_{ab}L_{cc}$
$+c_6^+ L_{ab}L_{ac}L_{bc} + c_7^+ R_{ambm}L_{ab} + c_8^+ R_{abcb}L_{ac} + c_9^+ R_{amam}L_{bb} + c_{10}^+\tau_{;m}$
$+c_{11}^+ L_{aa:bb} + c_{12}^+ L_{ab:ab} + c_{13}^+ S L_{aa}L_{bb} + c_{14}^+ S L_{ab}L_{ab} + c_{15}^+ S R_{amam} + c_{16}^+ S\tau$
$+c_{17}^+ S^2 L_{aa} + c_{18}^+ S^3 + c_{19}^+ S_{:aa} + c_{20}^+(SE^D + E^D S))\mathcal{B}_S^+ f_1, \widetilde{\mathcal{B}}_S^+ f_2\rangle$
$+\langle(d_1^+ L_{aa} + d_4^+ S)(\mathcal{B}_S^+ f_1)_{:b}, (\widetilde{\mathcal{B}}_S^+ f_2)_{:b}\rangle + d_2^+ L_{ab}\langle(\mathcal{B}_S^+ f_1)_{:a}, (\widetilde{\mathcal{B}}_S^+ f_2)_{:b}\rangle$
$+d_3^+(\langle\Omega^D_{am}(\mathcal{B}_S^+ f_1)_{:a}, \widetilde{\mathcal{B}}_S^+ f_2\rangle - \langle\Omega^D_{am}\mathcal{B}_S^+ f_1, (\widetilde{\mathcal{B}}_S^+ f_2)_{:a}\rangle)\}\,dy.$

PROOF. Define $\mathcal{E}$ as the remainder in the decomposition:

$$\beta_4(f_1, f_2, D, \mathcal{B}^-) = \frac{1}{2}\int_M \langle Df_1, \widetilde{D}f_2\rangle\, dx + \int_{\partial M} \Big\{\frac{1}{2}\langle (Df_1)_{;m}, f_2\rangle \tag{3.4}$$

$$+ \frac{1}{2}\langle f_1, (\widetilde{D}f_2)_{;m}\rangle - \frac{1}{4}\langle L_{aa}Df_1, f_2\rangle - \frac{1}{4}\langle L_{aa}f_1, \widetilde{D}f_2\rangle \tag{3.5}$$

$$+ \mathcal{E}(f_1, f_2, D, \mathcal{B}^-)\Big\}\, dy; \tag{3.6}$$

these constants are motivated by Lemma 2.1 (c). If $f_1|_{\partial M} = 0$, then

$$\begin{aligned} &\beta_4(f_1, f_2, D, \mathcal{B}^-) \\ &= -\frac{1}{2}\beta_2(Df_1, f_2, D, \mathcal{B}^-) = -\frac{1}{2}\beta_2(f_2, Df_1, \widetilde{D}, \widetilde{\mathcal{B}}^-) \\ &= \frac{1}{2}\int_M \langle Df_1, \widetilde{D}f_2\rangle\, dx + \int_{\partial M}\Big\{\frac{1}{2}\langle (Df_1)_{;m}, f_2\rangle \\ &\quad + \frac{1}{2}\langle f_1, (\widetilde{D}f_2)_{;m}\rangle - \frac{1}{4}\langle L_{aa}Df_1, f_2\rangle - \frac{1}{4}\langle L_{aa}f_1, \widetilde{D}f_2\rangle\Big\}\, dy. \end{aligned} \tag{3.7}$$

Consequently $\int_{\partial M}\mathcal{E}(f_1, f_2, D, \mathcal{B})\, dy = 0$ if $f_1|_{\partial M} = 0$. Since (3.0) and (3.0) are symmetric in f_1 and f_2, by Lemma 2.1 (a),

$$\int_{\partial M}\mathcal{E}(f_1, f_2, D, \mathcal{B}^-)\, dy = \int_{\partial M}\mathcal{E}(f_2, f_1, \widetilde{D}, \widetilde{\mathcal{B}}^-)\, dy. \tag{3.8}$$

This integral vanishes if either $f_1|_{\partial M} = 0$ or $f_2|_{\partial M} = 0$ so $\mathcal{E}$ involves only tangential derivatives of the f_i. We complete the proof by writing a Weyl basis for the space of such invariants; see (4.1) for some identities we have used. There is no invariant involving $\Omega_{am:a}$ since $\widetilde{\Omega}_{am:a} = -\Omega_{am:a}$. We omit the following invariants which might be thought to appear since their integrals are linear combinations of those appearing in the Lemma:

$$\begin{aligned} \{&L_{aa}(\langle f_{1:bb}, f_2\rangle + \langle f_1, f_{2:bb}\rangle),\ L_{ab}(\langle f_{1:ab}, f_2\rangle + \langle f_1, f_{2:ab}\rangle), \\ &R_{amab}\langle f_1, f_2\rangle_{:a}),\ L_{ab:a}\langle f_1, f_2\rangle_{:b},\ L_{aa:b}\langle f_1, f_2\rangle_{:b}\}. \end{aligned} \tag{3.9}$$

This proves (a), the proof of (b) is similar. □

We complete the proof of Theorems 1.1 and 1.2 by evaluating the constants of Lemma 3.1 in a series of steps. We use Lemmas from §2 and we adopt the notation of those Lemmas during their application. We begin by using Lemma 2.3 to evaluate certain constants appearing in β_4^- and β_6^+ in terms of constants appearing in β_2^-, β_2^+, and β_4^+ which have already been computed previously. This is the inductive step in the calculation.

STEP 3.2. (a) $c_2^- = c_3^- = d_1^- = 0$.
(b) $d_1^+ = -\frac{1}{12}$, $d_4^+ = -\frac{1}{6}$, $c_2^+ = \frac{1}{12}$, $c_3^+ = 0$, $c_{16}^+ = 0$, $c_{20}^+ = \frac{1}{12}$.

PROOF. We use Lemma 2.3. We equate the cross terms in L_{aa} in the following expressions to prove (a):

$$(3.10)\quad \beta_4(f_1,f_2,D,\mathcal{B}^-) = \int_{\partial M}\Big\{L_{aa}\langle\phi_1,\phi_2\rangle\Big\langle\Big(\Big(\frac{1}{2}+c_2^-\Big)E_2+c_3^-\tau+\Big(\frac{1}{2}-d_1^-\Big)\nabla^2)\psi_1,\psi_2\Big\rangle\,dy+\cdots$$

$$(3.11)\quad \beta_2(\phi_1,\phi_2,D_1,\mathcal{B}^-)\beta_2(\psi_1,\psi_2,D_2) = \int_{\partial M}\Big\{\frac{1}{2}L_{aa}\langle\phi_1,\phi_2\rangle\langle(E_2+\nabla^2)\psi_1,\psi_2\rangle\,dy+\cdots.$$

Similarly, we equate the cross terms in L_{aa} and S to prove (b):

$$(3.12)\quad \beta_6(f_1,f_2,D,\mathcal{B}_S^+) = \int_{\partial M}\Big\{L_{aa}\langle\mathcal{B}_S^+\phi_1,\widetilde{\mathcal{B}}_S^+\phi_2\rangle\Big\langle\Big(\Big(\frac{1}{6}+c_2^+\Big)E_2+c_3^+\tau+\Big(\frac{1}{6}-d_1^+\Big)\nabla^2\Big)\psi_1,\psi_2\Big\rangle + \langle S\mathcal{B}_S^+\phi_1,\widetilde{\mathcal{B}}_S^+\phi_2\rangle\Big\langle\Big(\Big(\frac{1}{3}+2c_{20}^+\Big)E_2+c_{16}^+\tau+\Big(\frac{1}{3}-d_4^+\Big)\nabla^2\Big)\psi_1,\psi_2\Big\rangle\Big\}\,dy+\cdots$$

$$(3.13)\quad \beta_4(\phi_1,\phi_2,D_1,\mathcal{B}_S^+)\beta_2(\psi_1,\psi_2,D_2) = \int_{\partial M}\Big\{\frac{1}{4}L_{aa}\langle\mathcal{B}_S^+\phi_1,\widetilde{\mathcal{B}}_S^+\phi_2\rangle\langle(E_2+\nabla^2)\psi_1,\psi_2\rangle + \frac{1}{2}\langle S\mathcal{B}_S^+\phi_1,\widetilde{\mathcal{B}}_S^+\phi_2\rangle\langle(E_2+\nabla^2)\psi_1,\psi_2\rangle\Big\}\,dy+\cdots.$$

□

In the remaining steps, we use relations between the constants in β_4^- and β_6^+ to determine the values of these constants in terms of values computed previously; the values computed in the inductive Step 2.3 form the starting point. We use Lemma 2.2 to relate Neumann and Dirichlet boundary conditions:

STEP 3.3. $c_1^-=\frac{1}{8}$, $c_1^+=\frac{1}{24}$, and $c_{18}^+=\frac{1}{6}$.

PROOF. By Lemma 2.2,

$$(3.14)\quad \beta_6(f_1,f_2,D_1,\mathcal{B}_S^+) = -\frac{1}{3}\beta_4(Af_1,Af_2,D_2,\mathcal{B}^-).$$

Let $b_i=\partial_\nu^i b$. We expand β_6^+ and β_4^- in terms of b and equate coefficients:

$$(3.15)\quad \int_{\partial M}\varepsilon\Big\{c_1^+(b_2-2b_1b)+\frac{1}{6}(b_1b-b^3)+c_{18}^+b^3\Big\}\langle Af_1,Af_2\rangle\,dy = -\frac{1}{3}\int_{\partial M}\varepsilon c_1^-(-b_2-2b_1b)\langle Af_1,Af_2\rangle\,dy.$$

□

At this point, the computation of Dirichlet and Neumann boundary conditions decouples and each calculation is independent. In the next three steps, we use Lemma 2.4 to relate $\beta_\nu^\pm$ for $M = [r_0, r_1] \times M_1$ to $\beta_\nu^\pm$ for $[r_0, r_1]$.

STEP 3.4.(a) $c_4^- = 0$, $c_5^- = -\frac{1}{16}$, $c_6^- = \frac{1}{8}$.
(b) $c_4^+ = 0$, $c_5^+ = \frac{1}{48}$, $c_6^+ = \frac{1}{24}$, $c_{13}^+ = \frac{1}{12}$, $c_{14}^+ = \frac{1}{12}$, $c_{17}^+ = \frac{1}{4}$.

PROOF. We use Lemma 2.4. Let $S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$ and $M = [r_0, r_1] \times S^n$. Identify M with the annulus in $\mathbf{R}^{n+1}$ and give M the flat Euclidean metric

$$ds_M^2 = dr^2 + r^2 ds_{S^n}^2.$$

Then the Laplacian D_0^M is given by:

$$D_0^M = D + r^{-2} D_0^{S^n} \tag{3.16}$$

where $D = -\partial_r^2 - nr^{-1}\partial_r$. We use (4.3) and (4.5) to compute:

$$\begin{aligned} &\beta_4(1, 1, D_0^M, \mathcal{B}^-) = \{-c_4^- n^3 - c_5^- n^2 - c_6^- n\} \int_{\partial M} \varepsilon r^{n-3}\, dy, \\ &\beta_4(1, r^n, D, \mathcal{B}^-) \operatorname{vol}(S^n) = \frac{1}{8}\Big(\frac{1}{2}n^2 - n\Big) \int_{\partial[r_0, r_1]} \varepsilon r^{n-3}\, dy_1 \operatorname{vol}(S^n) \\ &\qquad = \frac{1}{8}\Big(\frac{1}{2}n^2 - n\Big) \int_{\partial M} \varepsilon r^{n-3}\, dy. \end{aligned} \tag{3.17}$$

We equate these two expressions to prove (a). The proof of (b) is similar. Let

$$\mathcal{B}_S^+ = \varepsilon(\partial_r + sr^{-1}) f|_{\partial M} \tag{3.18}$$

so $S(D_0^M) = \varepsilon s r^{-1}$ and $S(D) = \varepsilon\Big(s - \frac{1}{2}n\Big) r^{-1}$; $\widetilde{\mathcal{B}}_{S(D)}^+(r^n) = sr^{n-1}$. Then

$$\begin{aligned} \beta_6(1, 1, D_0^M, \mathcal{B}_S^+) =& s^2\Big\{ -c_4^+ n^3 - c_5^+ n^2 - c_6^+ n + c_{13}^+ s n^2 \\ &+ c_{14}^+ s n - c_{17}^+ s^2 n + \frac{1}{6}s^3\Big\} \int_{\partial M} \varepsilon r^{n-5}\, dy, \\ \beta_6(1, r^n, D, \mathcal{B}_S^+) \operatorname{vol}(S^n) =& s^2\Big\{\frac{1}{24}\Big(\frac{1}{2}n^2 - n\Big) + \frac{1}{6}\Big(s - \frac{1}{2}n\Big)^3 \\ &+ \frac{1}{6}\Big(s - \frac{1}{2}n\Big)\Big(\frac{1}{2}n - \frac{1}{4}n^2\Big)\Big\} \int_{\partial M} \varepsilon r^{n-5}\, dy. \end{aligned} \tag{3.19}$$

□

STEP 3.5.(a) $c_7^- = -\frac{1}{16}$, $c_8^- = \frac{1}{16}$, $c_9^- = 0$, $c_{10}^- = \frac{1}{32}$.
(b) $c_7^+ = -\frac{1}{48}$, $c_8^+ = \frac{1}{48}$ $c_9^+ = -\frac{1}{24}$, $c_{10}^+ = \frac{1}{96}$, $c_{15}^+ = -\frac{1}{12}$.

PROOF. We apply Lemma 2.4 again. Let $M = [r_0, r_1] \times T^n$ where T^n is the flat torus with periodic parameters y_a. Let $\sigma|_{\partial M} = 0$ and let $\sigma_k = \partial_r^k \sigma$. Give M the metric:

$$ds_M^2 = dr^2 + e^{2\sigma}(dy_1^2 + \dots + dy_n^2); \tag{3.20}$$

$D_0^M = D + e^{-2\sigma} D_0^{S^n}$ for $D = -\partial_r^2 - n\sigma_1 \partial_r$. We use (4.4) and (4.5) to compute:

$$\begin{aligned} \beta_4(1, 1, D_0^M, \mathcal{B}^-) = \int_{\partial M} \varepsilon\Big\{ \Big(\frac{1}{16}n^2 - \frac{1}{8}n - nc_7^- + (n - n^2)c_8^- - n^2 c_9^-\Big)\sigma_1^3 \\ + (-nc_7^- - n^2 c_9^- - 2(n^2 + n)c_{10}^-)\sigma_1\sigma_2 - 2nc_{10}^-\sigma_3 \Big\}\, dy, \end{aligned} \tag{3.21}$$

$$\beta_4(1, e^{n\sigma}, D, \mathcal{B}^-)\mathrm{vol}(T^n) = \frac{1}{8}\int_{\partial M} \varepsilon\Big(-\frac{1}{2}n\sigma_3 - \frac{1}{2}n^2\sigma_1\sigma_2\Big)\, dy.$$

We equate these expressions to prove (a). To prove (b), we let

$$\mathcal{B}_S^+ f = \varepsilon(\partial_r + s)f|_{\partial M}; \tag{3.22}$$

$S(D_0^M) = \varepsilon s$ and $S(D) = \varepsilon\Big(s - \frac{1}{2}n\sigma_1\Big)$. We equate:

$$\begin{aligned} &\beta_6(1, 1, D_0^M, \mathcal{B}_S^+) \\ &= \int_{\partial M} \varepsilon\Big\{ \Big(-\frac{1}{48}n^2 - \frac{1}{24}n - nc_7^+ - n^2 c_8^+ + nc_8^+ - n^2 c_9^+\Big)\sigma_1^3 \\ &+ (-nc_7^+ - n^2 c_9^+ - 2n^2 c_{10}^+ - 2nc_{10}^+)\sigma_1\sigma_2 - 2nc_{10}^+\sigma_3 \\ &+ \Big(\frac{1}{12}n^2 + \frac{1}{12}n + nc_{15}^+\Big)s\sigma_1^2 + nc_{15}^+ s\sigma_2 - \frac{1}{4}ns^2\sigma_1 + \frac{1}{6}s^3 \Big\}\, dy, \end{aligned} \tag{3.23}$$

$$\begin{aligned} \beta_6(1, e^{n\sigma}, D, \mathcal{B}_S^+)\ \mathrm{vol}(T^n) = \int_{\partial M} \varepsilon\Big\{ \frac{1}{24}\Big(-\frac{1}{2}n\sigma_3 - \frac{1}{2}n^2\sigma_1\sigma_2\Big) \\ + \frac{1}{6}\Big(-\frac{1}{2}n\sigma_2 - \frac{1}{4}n^2\sigma_1^2\Big)\Big(s - \frac{1}{2}n\sigma_2\Big) + \frac{1}{6}\Big(s - \frac{1}{2}n\sigma_1\Big)^3 \Big\}\, dy. \end{aligned}$$

□

STEP 3.6. (a) $d_2^- = -\frac{1}{4}$, $d_3^- = -\frac{1}{8}$, $d_2^+ = -\frac{1}{12}$, $d_3^+ = -\frac{1}{24}$. (b) $c_{19}^+ = \frac{1}{12}$.

PROOF. We use Lemma 2.4 with an auxiliary first order term. Let M be as in Step 3.5. Let $\hat{\sigma}|_{\partial M} = 1$. Define:

$$D = -(\partial_r^2 + n\sigma_1\partial_r + e^{-2\sigma}(\Sigma_a \partial_{y_a}^2 + 2\hat{\sigma}h_a\partial_{y_a})). \tag{3.24}$$

Let ψ_2 and h_a be constant. We use (4.4). Since $\beta_\nu(1, e^{n\sigma_1}, D_1, \mathcal{B})$ does not involve h, we set the terms which involve h to zero to prove (a):

$$\begin{aligned} 0 &= \int_{\partial M} \varepsilon\Big\{\Big(\frac{1}{4} + d_2^-\Big)\sigma_1 + \Big(-\frac{1}{4} - 2d_3^-\Big)\widehat{\sigma}_1\Big\} h_a^2 \psi_2 \, dy, \\ 0 &= \int_{\partial M} \varepsilon\Big\{\Big(\frac{1}{12} + d_2^+\Big)\sigma_1 + \Big(-\frac{1}{12} - 2d_3^+\Big)\widehat{\sigma}_1\Big\} h_a^2 \psi_2 \, dy. \end{aligned} \tag{3.25}$$

To prove (b), we consider vector valued operators. Let $V = M \times \mathbf{C}^2$ and let $D = D_0^M - 2h_a\partial_a$. Let f_i and h_a be constant, and let S be locally constant. We use (4.6) to see:

$$\begin{aligned} 0 = \int_{\partial M} \Big\langle \Big\{ \Big(-\frac{1}{12} + c_{19}^+\Big) h_a h_a S + \Big(-\frac{1}{12} + c_{19}^+\Big) S h_a h_a \\ + \Big(\frac{1}{6} - 2c_{19}^+\Big) h_a S h_a \Big\} f_1, f_2 \Big\rangle \, dy. \end{aligned} \tag{3.26}$$

□

We use Lemma 2.5 to obtain the remaining undetermined coefficients.

STEP 3.7.(a) $c_{11}^- = 0$, $c_{12}^- = \frac{1}{16}$.
(b) $c_{11}^+ = 0$, $c_{12}^+ = \frac{1}{48}$.

PROOF. Let $M = [r_0, r_1] \times T^n$. Let $\sigma = \sigma(r)$ and $\rho = \rho(y)$. We assume $\sigma_k = 0$ near $r = r_i$ for $k > 1$. Let δ be a small parameter. Give M the warped product metric:

$$ds^2 = dr^2 + (1 + \delta\sigma\rho)^2(dy_1^2 + \dots + dy_n^2). \tag{3.27}$$

and let $D = D_1 + D_2$ for $D_1 = -\partial_r^2$ and $D_2 = -(1 + \delta\sigma\rho)^{-2}\Sigma_a \partial_{y_a}^2$. Let $f_1 = 1$ and let $f_2 = \psi_2(y)$. Thus $\phi_1 = \phi_2 = \psi_1 = 1$. We use Lemma 2.5 to express:

$$\beta_\nu(f_1, f_2, D, \mathcal{B}) = \Sigma_k \tbinom{m_2}{k} \delta^k \beta_\nu(\phi_1, \sigma^k\phi_2, D_1, \mathcal{B}) \int_{M_2} \langle \psi_1, \rho^k \psi_2 \rangle \, dx_2. \tag{3.28}$$

The coefficient of $\int_{\partial M} \delta\varepsilon\sigma_1\psi_2\partial_a^2\rho \, dy$ in $\beta_4(1, \psi_2, D, \mathcal{B}^-)$ is zero since (3.28) contains no terms involving derivatives of ψ_2 and ρ; this term is nonzero since ψ_2 is arbitrary. We use (4.7) and Lemma 3.1 prove (a) by showing that

$$0 = \Big(\frac{1}{4}n - \frac{1}{4}n + \frac{1}{16}(n-2) + \frac{1}{16}(1-n) + \frac{1}{8} - c_{11}^- n - c_{12}^-\Big). \tag{3.29}$$

We take boundary conditions $\mathcal{B}_S^+ f = \varepsilon(\partial_r + s) f|_{\partial M} = 0$ where s is locally constant on ∂M. Since

$$\omega_m^D = -\frac{1}{2}\varepsilon n \delta\sigma_1\rho + O(\delta^2), \tag{3.30}$$

$S = s + \frac{1}{2}\varepsilon n\delta\sigma_1\rho$. The same argument shows the coefficient of

$$\int_{\partial M} \delta\varepsilon\sigma_1\psi_2\delta_a^2\rho s^2\, dy \tag{3.31}$$

in $\beta_6(1,\psi_2,D,\mathcal{B}_S^+)$ is zero. We complete the proof by showing that

$$0 = \Big(-\frac{1}{12}n + \frac{1}{48}(n-2) + \frac{1}{48}(1-n) + \frac{1}{12}n + \frac{1}{24} - nc_{11}^+ - c_{12}^+\Big). \tag{3.32}$$

□

This completes the proof of all the assertions in the paper.

4. Appendix

We summarize for convenient reference some calculations that are immediate from the definitions given.

(4.1) We have the following identities:

1) $\Gamma_{\nu\mu}{}^{\sigma} = \frac{1}{2}g^{\sigma\varepsilon}(\partial_\nu g_{\mu\varepsilon} + \partial_\mu g_{\nu\varepsilon} - \partial_\varepsilon g_{\nu\mu})$,

2) $R_{\nu_1\nu_2\nu_3}{}^{\nu_4} = \partial_{\nu_1}\Gamma_{\nu_2\nu_3}{}^{\nu_4} - \partial_{\nu_2}\Gamma_{\nu_1\nu_3}{}^{\nu_4} + \Gamma_{\nu_1\mu_2}{}^{\nu_4}\Gamma_{\nu_2\nu_3}{}^{\mu_2} - \Gamma_{\nu_2\mu_2}{}^{\nu_4}\Gamma_{\nu_1\nu_3}{}^{\mu_2}$,

3) $D_0 = -g^{-1}\partial_\nu g g^{\nu\mu}\partial_\mu$ for $g = \det(g_{\nu\mu})^{1/2}$.

(4.2) If $D = -(g^{ij}\partial_i\partial_j + p^k\partial_k + q)$, then:

1) $\omega_\nu^D = \frac{1}{2}g_{\nu\mu}(P^\mu + g^{\sigma\varepsilon}\Gamma_{\sigma\varepsilon}{}^{\mu})$, $\Omega_{\nu\mu}^D = \partial_\nu\omega_\mu^D - \partial_\mu\omega_\nu^D + \omega_\nu^D\omega_\mu^D - \omega_\mu^D\omega_\nu^D$,

2) $E^D = q - g^{\nu\mu}(\partial_\nu\omega_\mu^D + \omega_\nu^D\omega_\mu^D - \omega_\sigma^D\Gamma_{\nu\mu}{}^{\sigma})$, $\omega^{D_0} = \Omega^{D_0} = E^{D_0} = 0$.

3) $L_{\alpha\beta} = (\partial_m, \nabla_{\partial_\alpha}\partial_\beta) = \Gamma_{\alpha\beta}{}^{m} = -\frac{1}{2}\partial_m g_{\alpha\beta}$, $f_{;ab} = f_{:ab} - L_{ab}f_{;m}$,

4) $R_{abcN} = L_{bc:a} - L_{ac:b}$, $R_{abab;m} = 2R_{abmb;a} = p(L_{..:..}, L_{..}, R_{....})$.

(4.3) Let $ds^2 = \partial_r^2 + r^2(ds_\theta^2)$ on $[r_0, r_1]\times S^n$. On ∂M :

1) $L_{aa}L_{bb}L_{cc} = -\varepsilon r^{-3}n^3$, $L_{ab}L_{ab}L_{cc} = -\varepsilon r^{-3}n^2$, $L_{ab}L_{ac}L_{bc} = -\varepsilon r^{-3}n$,

2) $R_{ambm}L_{ab} = R_{abcb}L_{ac} = R_{amam}L_{bb} = \tau_{;m} = 0$.

(4.4) Let $ds^2 = \partial_r^2 + e^{2\sigma(r)}(dy_1^2 + \dots + dy_n^2)$ on $[r_0, r_1]\times T^n$.

Let $\sigma|_{\partial M} = 0$, and let $\partial_r^k\sigma = \sigma_k$. On ∂M :

1) $\Gamma_{am}{}^{\beta} = \Gamma_{m\alpha}{}^{\beta} = \sigma_1\delta_\alpha^\beta$, $\Gamma_{\alpha\beta}{}^{m} = -\sigma_1 e^{2\sigma}\delta_{\alpha\beta}$, and $\Gamma = 0$ otherwise,

2) $\tau = -2n\sigma_2 - n\sigma_1^2 - n^2\sigma_1^2$, $L_{aa}L_{bb} = n^2\sigma_1^2$, $L_{ab}L_{ab} = n\sigma_1^2$,

3) $R_{amam} = n\sigma_1^2 + n\sigma_2$, $\tau_{;m} = -2n^2\sigma_1\sigma_2 - 2n\sigma_1\sigma_2 - 2n\sigma_3$,

4) $L_{aa}L_{bb}L_{cc} = -n^3\sigma_1^3$, $L_{ab}L_{ab}L_{cc} = -n^2\sigma_1^3$, $L_{ab}L_{bc}L_{ac} = -n\sigma_1^3$,

5) $R_{ambm}L_{ab} = -n\sigma_1^3 - n\sigma_1\sigma_2$, $R_{abcb}L_{ac} = -n^2\sigma_1^3 + n\sigma_1^3$,

6) $R_{amam}L_{bb} = -n^2\sigma_1^3 - n^2\sigma_1\sigma_2$.

Let $D = D_0^M - 2e^{2\sigma}\widehat{\sigma}h_a\partial_a$ where h is constant and $\widehat{\sigma}|_{\partial M} = 1$. Then:

7) $\omega_m^D = 0$, $\omega_a^D = \widehat{\sigma}h_a$, $\Omega_{am}^D = -\widehat{\sigma}_1 h_a$,

8) $E^D = -e^{-2\sigma}\widehat{\sigma}^2 h_a^2$, $E_{;m}^D|_{\partial M} = e^{-2\sigma}(-2\widehat{\sigma}_1 h_a^2 + 2\sigma_1 h_a^2)$.

(4.5) If $D = -\partial_r^2 - n\sigma_1(r)\partial_r$,

1) $\varepsilon f_{1;m} = (\partial_r + \frac{1}{2}n\sigma_1(r))f_1$, $\varepsilon f_{2;m} = (\partial_r - \frac{1}{2}n\sigma_1(r))f_2$,

2) $E^D = -\frac{1}{2}n\sigma_2 - \frac{1}{4}n^2\sigma_1^2$, $E_{;m}^D = \varepsilon(-\frac{1}{2}n\sigma_3 - \frac{1}{2}n^2\sigma_1\sigma_2)$,

3) $D(1) = 0$, and $\widetilde{D}(e^{n\sigma}) = 0$.

(4.6) Let $M = [r_0, r_1] \times T^n$, let $D = -\partial_r^2 - \partial_a^2 - 2h_a\partial_a$ on $C^\infty(M \times \mathbf{C}^2)$.

Let $\{h_a, f_i, S\}$ be (locally) constant. Then:

1) $E^D = -h_a h_a$, $\langle Sf_{1:a}, f_{2:a}\rangle = \langle -Sh_a f_1, h_a f_2\rangle$,

2) and $S_{:aa} = h_a h_a S - 2h_a S h_a + S h_a h_a$.

(4.7) Let $M = [r_0, r_1] \times T^n$, let $\sigma|_{\partial M} = 0$ and $\sigma_k|_{\partial M} = 0$ for $k > 1$.

Let $ds^2 = dr^2 + (1 + 2\delta\sigma\rho + O(\delta^2))(dy_1^2 + \ldots + dy_n^2)$, $\mathcal{B}_S^+ = \varepsilon(\partial_r + s)$ and $D = -(\partial_r^2 + (1 - 2\delta\sigma\rho)\partial_a^2) + O(\delta^2)$. Then:

1) $L_{aa}L_{bb}L_{cc} = O(\delta^3)$, $L_{ab}L_{ab}L_{cc} = O(\delta^3)$, $L_{ab}L_{ac}L_{bc} = O(\delta^3)$,

2) $R_{ambm}L_{ab} = O(\delta^3)$, $R_{abcb}L_{ac} = O(\delta^3)$, $R_{amam}L_{bb} = O(\delta^3)$,

3) $\tau_{;m} = 2\delta(1-n)\sigma_1\partial_a^2\rho + O(\delta^2)$, $L_{aa:bb} = -n\sigma_1\rho_{:aa} + O(\delta^2)$,

4) $L_{ab:ab} = -\sigma_1\rho_{:aa} + O(\delta^2)$, $\omega_a^D = \frac{1}{2}(2-n)\delta\sigma\partial_a(\rho) + O(\delta^2)$,

5) $\omega_m^D = -\frac{1}{2}n\delta\sigma_1\rho + O(\delta)^2$, $S = s + \frac{1}{2}n\delta\sigma_1\rho + O(\delta^2)$, $S_{:aa} = \frac{1}{2}n\delta\sigma_1\rho_{:aa}$,

6) $E^D = \frac{1}{2}(n-2)\delta\sigma\partial_a^2(\rho)$, $E_{;m}^D = \frac{1}{2}(n-2)\delta\sigma_1\partial_a^2(\rho) + O(\delta)^2$,

7) $\Omega_{am}^D = -\delta\sigma_1\partial_a(\rho) + O(\delta^2)$, $(\widetilde{D}\psi)_{;m} = \frac{1}{2}n\delta\sigma_1\rho\partial_a^2\psi + O(\delta^2)$.

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