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ON EXISTENCE OF SOLUTIONS OF MIXED PROBLEMS FOR PARABOLIC SYSTEMS

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(Submitted by M. Burnat)

Dedicated to the memory of Juliusz Schauder

1. Introduction

In this paper we consider the following initial boundary value problem for a system of quasilinear parabolic equations

$$\partial_t b^j(u) - \nabla \cdot a^j(x,t,u,\nabla u) = f^j(x,t,u,\nabla u) \quad \text{in } Q_T := \Omega \times (0,T),\ j = 1,\dots,n. \tag{1.1}$$

$$a^j(x,t,u,\nabla u)\cdot \nu(x) = g^j(x,t,u) \qquad \text{on } S_T := \partial\Omega \times (0,T),\ j = 1,\dots,n. \tag{1.2}$$

$$u(x,0) = u_0(x) \qquad \text{on } \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with smooth boundary $\partial = \Omega$, $\nu(x) = (\nu_1,\dots,\nu_N)$ denotes the outer unit normal to $\partial\Omega$, $u = (u^1,\dots,u^n)$, $n \geq 1$, $\nabla u = (\nabla u^1,\dots,\nabla u^n)$, $\nabla = \operatorname{grad}_x$.

This paper is motivated by results of Filo and Kačur [8]. The paper [8] concerns the existence of a variational solution to problem (1.1) – (1.3) with f^j, $j = 1,\dots,n$, independent of ∇u. In contrast to [8] we consider here the case when f^j are depending not only on x, t and u but on ∇u, as well. We assume here some structure and monotonicity conditions (see assumptions (H1)–(H6) in the next section) among which the conditions concerning functions b^j ((H1) and (H2)), function g ((H3)) and the structure condition ((H4)) imposed on a^j are the same as the corresponding conditions from [8]. The other assumptions of [8], i.e..:

1° the monotonicity condition

$$\sum_{j=1}^{n}(a^j(x,t,z,q_1)-a^j(x,t,z,q_2))(q_1-q_2)\geq 0$$

$\forall(x,t)\in Q_T$, $\forall z\in\mathbb{R}^n$ and $\forall q_i=(q_i^1,\dots,q_i^n)$, $i=1,2$, $q_i^j\in\mathbb{R}^n$ and the coerciveness condition

$$\sum_{j=1}^{n}a^j(x,t,z,q)\cdot q^j\geq c_1|q|^{r+1}-c_2;$$

2° the structure condition

$$|f(x,t,z)|\leq c(1+|z|^p),\qquad (p>0),$$

are replaced in our paper by

1* the strict monotonicity condition

$$\sum_{j=1}^{n}(a^j(x,t,z,q_1)-a^j(x,t,z,q_2))(q_1^j-q_2^j)\geq c|q_1-q_2|^{r+1};$$

2* the structure condition

$$|f(x,t,z,q)|\leq c(1+|z|^p+|q|^s),\qquad (s>0).$$

The paper is divided into four sections. Section 2 contains notation used in paper, and Section 3 is devoted to the existence of a variational solution to problem (1.1)–(1.3). Section 3 consists of four parts. Part 3.1 contains assumptions (H1)–(H6) which have been presented above. In Part 3.2 the definition of a variational solution of (1.1)–(1.3) and the existence theorems are formulated. We admit the same assumptions on p and α (α is connected with the growth of g (see (H6))) as in [8], both in Theorem 1 which is referred to the local existence of solution of (1.1)–(1.3) and in Theorem 2 concerning the global existence of a solution. The restrictions imposed on p and α follow from the interpolation inequalities proved in [8]. Moreover, we assume an additional condition associated with the growth of f, i.e. $s<\max\left\{\frac{(r+1)m}{m+1},\frac{r(N+m+1)m+1}{N+m+1}\right\}$ (assumption (iii) of Theorem 1 and Theorem 2).

In Part 3.3 we introduce an auxiliary problem (see problem (3.1)–(3.3)) which is used to prove Theorems 1 and 2. We prove the existence of a variational solution of (3.1)–(3.3) applying the methods from the papers of Alt and Luckhaus [2] and Kačur [10].

Part 3.4 contains the proofs of Theorems 1 and 2. In the proofs the methods of [2], [8] and [10] are used.

Finally, Section 4 concerns the existence of a variational solution to problem (1.1)–(1.3) in the case when $b = \mathrm{id}$. We formulate there Theorems 3 and 4 analogous to Theorems 1 and 2 of Section 3.

Quasilinear parabolic systems in the case $b = \mathrm{id}$ under general nonlinear boundary conditions were considered in papers [1], [3], [4], [5] and [9]. In [1] P. Acquistapace and B. Terreni prove some results on local in time existence of continuously differentiable solutions of such problems by using $W^{2,p}$-estimates (where $p > N$). The existence of the classical local solution is also proved by H. Amann in [3] and by M. Giaquinta and G. Modica in [9]. H. Amann uses in his paper [3] semigroup methods, while in [9] methodes based on Schauder type estimates are used. Paper [4] contains the results concerning both classical and weak solutions of semilinear parabolic systems under nonlinear boundary conditions. At last, in [5] some recent results on theory of linear and quasilinear elliptic and parabolic systems with nonhomogenous boundary conditions are described.

2. Notation

We use the same notation as in [4]. In the sequel we denote by b, a^j, $j = 1,\dots,n$, a, f, g the vectors $(b^1,\dots,b^n)$, $(a_1^j,\dots,a_N^j)$, $(a^1,\dots,a^n)$, $(f^1,\dots,f^n)$, $(g^1,\dots,g^n)$, respectively. Let X be whichever of the function spaces mentioned in this paper. We say that a function $v = (v^1,\dots,v^n)$ belongs to X if $\forall 1 \le i \le n$, $u^i \in X$. Next, we use the following notation: $b(z)z = \sum_{j=1}^{n} b^j(z)z_j$ for $z \in \mathbb{R}^n$; $a(u,\nabla u) := a(x,t,u,\nabla u)$, $f(u,\nabla u) := f(x,t,u,\nabla u)$, $g(u,\nabla u) := g(x,t,u,\nabla u)$; $\langle\cdot,\cdot\rangle$ — the duality between $V := W^1_{r+1}(\Omega)$ and V^*; $\int_{\partial\Omega} |v|^{\alpha+1} := \int_{\partial\Omega} |v(x)|^{\alpha+1}\,dS$; $\int_\Omega v(t)\phi(t) := \int_\Omega v(x,t)\phi(x,t)\,dx$, etc.

In this paper we also use the following interpolation inequality

$$\int_\Omega |v|^{p+1} \le \eta \|\nabla v\|^{r+1}_{L^{r+1}(\Omega)} + C\eta^{-\sigma}\left(\int_\Omega |v|^{m+1}\right)^{\gamma+1} \tag{2.1}$$

for any $v \in L^{m+1}(\Omega)$ with $\nabla v \in L^{r+1}(\Omega)$ and for any $0 < \eta < \infty$, where

$$0 < m \le p < \frac{r(N+m+1)+m+1}{N}, \qquad \gamma = \frac{(r+1)(p-m)}{r(N+m+1)+m+1-Np}$$

and

$$\sigma = \frac{N(p-m)}{r(N+m+1)+m+1-Np}.$$

Inequality (2.1) follows from the Gagliardo-Nirenberg inequality (see [7] and also [8], Prop. 1).

3. Existence of a solution of problem (1.1)–(1.3)

3.1. Assumptions

Now we introduce assumptions concerning the structure of problem (1.1)–(1.3). We assume the following properties:

(H1) There is a strictly convex C^1-function $\Phi : \mathbb{R}^n \to \mathbb{R}$, $\Phi(0) = 0$, $\nabla\Phi(0) = 0$ such that

$$b(z) = \nabla\Phi(z);$$

(H2) $B(z) := b(z)\cdot z - \Phi(z) = \int_0^1 (b(z) - b(sz))\cdot z\,ds$ satisfies

$$B(z) \geq c_1|z|^{m+1} - c_2 \qquad (m > 0),$$

where $c_1, c_2 > 0$ are constants.

(H3) $a^j : Q_T \times \mathbb{R}^n \times \mathbb{R}^{Nn} \to \mathbb{R}^N$ $j = 1,\dots,n$ are continuous (or satisfy Carathéodory conditions) and

$$\sum_{j=1}^n (a^j(x,t,z,q_1) - a^j(x,t,z,q_2))(q_1^j - q_2^j) \geq c|q_1 - q_2|^{r+1}$$

$\forall (x,t) \in Q_T$, $\forall z \in \mathbb{R}^n$ and $\forall q_i = (q_i^1,\dots,q_i^n)$, $i = 1,2$, where $q_i^j \in \mathbb{R}^N$ for $j = 1,\dots,n$, $r > 0$;

(H4) $\sum_{j=1}^n |a^j(x,t,z,q)| \leq c(1 + |z|^\vartheta + |q|^r)$, where $\vartheta = \max\{r, \frac{rp}{r+1}\}$, $p > 0$;

(H5) $f : Q_T \times \mathbb{R}^n \times \mathbb{R}^{Nn} \to \mathbb{R}^n$ is continuous (or satisfies Carathéodory condition) and

$$|f(x,t,z,q)| \leq c(1 + |z|^p + |q|^s), \qquad (s > 0);$$

(H6) $g : S_T \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous (or satisfies Carathéodory condition) and

$$|g(x,t,z)| \leq c(|z|^\alpha + 1), \qquad \alpha > 0.$$

In (H3)–(H6) $c > 0$ is a constant.

3.2. Definition of a variational solution of (1.1)–(1.3) and formulating of main theorems

At first, following [8] (see also [2] and [10]) we introduce the definition of a variational solution of problem (1.1)–(1.3).

DEFINITION 1. *A vector function* $u \in L^{r+1}(0,T;V) \cap L^{\infty}(0,T;L^{m+1}(\Omega))$ *is a variational solution of* (1.1)–(1.3) *on* Q_T *if and only if* $b(u) \in L^1(Q_T)$, $\partial_t b(u) \in L^{(r+1)/r}(0,T;V^*)$ *and*

(i) $\int_0^T \langle \partial_t B(u), v\rangle = -\iint_{Q_T}(b(u) - b(U_0))\partial_t v$
$\forall v \in L^{r+1}(0,T;V) \cap L^{\infty}(Q_T)$ *with* $\partial_t v \in L^{\infty}(Q_T)$, $v(T) = 0$;

(ii) $\int_0^T \langle \partial_t b(u), v\rangle + \iint_{Q_T} a(u,\nabla u)\nabla v - \iint_{S_T} g(u)v = \iint_{Q_T} f(u,\nabla u)v$,
$\forall v \in L^{r+1}(0,T;V) \cap L^{\infty}(0,T,L^{m+1}(\Omega))$ $(V = W^1_{r+1}(\Omega))$.

Now we shall formulate the main theorems which are analogous to Theorems 1 and 2 of [8].

THEOREM 1 (Local Existence). *Let* (H1)–(H6) *be satisfied. Moreover, let* $u_0 \in W^1_{r+1}(\Omega)$ *and* $u_0 b(u_0) \in L^1(\Omega)$. *Then there exists* $T^* \in (0,T]$ *such that problem* (1.1)–(1.3) *has a variational solution* u *on* Q_{T^*} *provided the following conditions are satisfied :*

(i) $0 < p < p^* := \max\left\{m, \frac{r(N+m+1)+m+1}{N}\right\}$;

(ii) $0 < \alpha < \frac{r(N+\min\{\alpha,m\}+1)}{N}$;

(iii) $0 < s < s^* := \max\left\{\frac{(r+1)m}{m+1}, \frac{r(N+m+1)+m+1}{N+m+1}\right\}$;

THEOREM 2 (Global Existence). *Let* (H1)–(H6) *be satisfied. Moreover, let* $u_0 \in W^1_{r+1}(\Omega)$ *and* $u_0 b(u_0) \in L^1(\Omega)$. *Then problem* (1.1)–(1.3) *has a variational solution on* Q_T *for any* $T > 0$ *provided the following conditions are satisfied:*

(i) $p \le m$ $(p < m$ *if* $p^* = m)$;

(ii) *either* $0 < \alpha < \min\{m, r\}$ *or* $0 < r < \alpha < \frac{r(N+\alpha+1)}{N}$ *and*

$$\alpha < \begin{cases} \frac{(m+1)r}{r+1} & \text{in the case } N = 1,\\ \frac{r(m(r+1)+m+1)}{r(r+1)+m+1} & \text{for } N = r+1,\\ \frac{N(r+m)-rm+1-(N(r-m)+mr-1)^2+4r(r+1)(m+1)}{2(N-r-1)} & \text{otherwise,} \end{cases}$$

(iii) $s \le \frac{(r+1)m}{m+1}$ $\left(s < \frac{(r+1)m}{m+1} \;\; \textit{if } s^* = \frac{(r+1)m}{m+1}\right)$.

3.3. An auxiliary problem

In order to prove the above theorems consider first the problem

$$\partial_t b^j(u_\varepsilon) - \nabla \cdot a^j_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) = f^j_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) \quad \text{in } Q_t, \tag{3.1}$$

$$a^j_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon)\nu(x) = g^j_\varepsilon(x,t,u_\varepsilon) \quad \text{on } S_T, \tag{3.2}$$

$$u_\varepsilon(x,0) = u_0(x) \quad \text{in } \Omega \tag{3.3}$$

where

$$\begin{aligned} a^j_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) &:= a^j(x,t,\zeta_\varepsilon(u_\varepsilon)u_\varepsilon,\nabla u_\varepsilon),\\ f_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) &:= f(x,t,\zeta_\varepsilon(u_\varepsilon)u_\varepsilon,\zeta_\varepsilon(\nabla u_\varepsilon)\nabla u_\varepsilon),\\ g_\varepsilon(x,t,u_\varepsilon) &:= g(x,t,\zeta_\varepsilon(u_\varepsilon)u_\varepsilon),\\ \zeta_\varepsilon(z) &:= \min\left\{1, \frac{1}{\varepsilon|z|}\right\}. \end{aligned} \tag{3.4}$$

The following lemma is true.

LEMMA 1. *Let* (H1)–(H6) *and assumption* (iii) *be satisfied. Then there exists a variational solution* u_ε *of* (3.1) – (3.3) *in* Q_T *for any* $0<\varepsilon\ll 1$.

PROOF. Similarly to [2] and [10] we prove the lemma under the assumption that a^j, f and g are independent of t. First, we replace $\partial_t b(u)$ by the backward difference quotient $\partial_t^{-h} b(u) = \frac{1}{h}[b(u(t)) - b(u(t-h))]$. Thus, instead of parabolic problem (3.1)–(3.3) we obtain an elliptic problem which we solve applying the Galerkin method. To do this we choose functions $e_1 \in W^1_{r+1}(\Omega)\cap L^{m+1}(\Omega)$ such that $\forall\lambda$, $e_1,\dots,e_\lambda$ are linearly independent and linear combinations of e_i are dense in $W^1_{r+1}(\Omega)\cap L^{m+1}(\Omega)$. As in [2] (see also [10]) we are looking for an approximate solution of (3.1) – (3.3) in the form

$$u_{h\lambda}(x,t) = \sum_{i=1}^{\lambda} \alpha_{h\lambda i}(t)e_i(x)$$

with $\alpha_{h\lambda i}\in L^\infty((0,T))$, where $u_{h\lambda}(x,t)$ satisfies the equality

$$\begin{aligned} S_{h\lambda}(u_{h\lambda},v) := &\int_\Omega \partial_t^{-h} b(u_{h\lambda}(t))v\\ &+ \int_\Omega a_\varepsilon(u_{h\lambda},\nabla u_{h\lambda})\nabla v - \int_{\partial\Omega} g_\varepsilon(u_{h\lambda})v\\ &- \int_\Omega f_\varepsilon(u_{h\lambda},\nabla u_{h\lambda})v = 0 \end{aligned} \tag{3.5}$$

for almost all $t \in (0,T)$ and for all $v \in V_\lambda := \operatorname{span}\{e_1, \dots, e_\lambda\}$. In (3.5) the initial data are given by

$$u_{h\lambda}(t) := u_h^0(t) \qquad \text{for } -h < t < 0, \tag{3.6}$$

$$u_h^0(t) := \min\left\{1, \frac{1}{h|u_0|}\right\}u_0. \tag{3.7}$$

For simplicity we assume that T/h is integer. From (3.5) we conclude that $u_{h\lambda}(t)$ can be determined inductively for $t \in ((k-1)h, kh)$ and $\alpha_{h\lambda}(t)$ are constants on $((k-1)h, kh)$.

Now we prove the existence of $u_{h\lambda}(t)$. To do this assume that $u_{h\lambda}(t)$ is known in $(0,(k-1)h)$. We must prove the existence of $u_{h\lambda}(t)$ in $(0,kh)$, so we must determine $\alpha = (\alpha_i)_{i=1,\dots,\lambda}$ for $t \in (0,kh)$. Denote $\phi = \sum_{i=1}^{\lambda} \alpha_i e_i$ and consider a continuous mapping $J_{h\lambda} : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ such that $J_{h\lambda}(\alpha) = (S_{h\lambda}(\phi, e_i), i = 1, \dots, \lambda)$. Using (3.5) we obtain

$$\begin{aligned} J_{h\lambda}(\alpha)\alpha &= \sum_{i=1}^{\lambda} S_{h\lambda}(\phi, e_i)\alpha_i = S_{h\lambda}(\phi,\phi) \\ &= \int_\Omega \partial_t^{-h} b(\phi)\phi + \int_\Omega a_\varepsilon(\phi, \nabla\phi)\nabla\phi - \int_{\partial\Omega} g_\varepsilon(\phi)\phi - \int_\Omega f_\varepsilon(\phi,\nabla\phi)\phi. \end{aligned} \tag{3.8}$$

Applying in (3.8) assumption (H3) we get

$$\begin{aligned} J_{h\lambda}(\alpha)\alpha \geq \frac{1}{h}\int_\Omega (b(\phi(t)) - b(\phi(t-h)))\phi + c\int_\Omega |\nabla\phi|^{r+1} \\ - \int_\Omega |a_\varepsilon(\phi,0)||\nabla\phi| - \int_{\partial\Omega} |g_\varepsilon(\phi)||\phi| - \int_\Omega |f_\varepsilon(\phi,\nabla\phi)||\phi|. \end{aligned} \tag{3.9}$$

Using (H4), (3.3), (3.4) and the Young inequality we have

$$\int_\Omega |a_\varepsilon(\phi,0)||\nabla\phi| \leq c\left(1 + \left(\frac{1}{\varepsilon}\right)^\vartheta\right)\int_\Omega |\nabla\phi| \leq \eta\int_\Omega |\nabla\phi|^{r+1} + C(\varepsilon,\eta). \tag{3.10}$$

Next, using (H6) and the Young inequality yields

$$\begin{aligned} \int_{\partial\Omega} |g_\varepsilon(\phi)||\phi| &\leq c\left(1 + \left(\frac{1}{\varepsilon}\right)^\alpha\right)\int_{\partial\Omega} |\phi| \\ &\leq \eta\int_{\partial\Omega} |\phi|^{1+\sigma} + C(\varepsilon,\eta), \quad 0 < \sigma < \min\{r, m\}. \end{aligned} \tag{3.11}$$

Applying now Remark 2 (p. 22) from [8] we get

$$
\begin{aligned}
(3.12)\qquad \int_{\partial\Omega} |\phi|^{1+\sigma} &\le \eta \int_\Omega |\nabla\phi|^{1+\sigma} + C(\eta)\int_\Omega |\phi|^{1+\sigma} \\
&\le \eta_1 \int_\Omega |\nabla\phi|^{1+r} + C(\eta_1)\int_\Omega |\phi|^{1+m} + C(\eta_1).
\end{aligned}
$$

Thus, by (3.11) and (3.12) we have

$$
(3.13)\qquad \int_{\partial\Omega} |g_\varepsilon(\phi)||\phi| \le \eta_2 \int_\Omega |\nabla\phi|^{1+r} + C(\eta_2)\int_\Omega |\phi|^{1+m} + C(\varepsilon,\eta_2).
$$

Now, using (H5) and the Young inequality we obtain

$$
(3.14)\qquad \int_\Omega |f_\varepsilon(\phi,\nabla\phi)||\phi| \le c\bigg(1+\bigg(\frac{1}{\varepsilon}\bigg)^p+\bigg(\frac{1}{\varepsilon}\bigg)^s\bigg)\int_\Omega|\phi| \le \eta\int_\Omega|\phi|^{m+1}+C(\varepsilon,\eta).
$$

Next, using the property (see [2]):

$$
(3.15)\qquad B(z_0)-B(z)\le (b(z_0)-b(z))z_0
$$

and (H2) we have

$$
\begin{aligned}
(3.16)\qquad &\frac{1}{h}\int_\Omega (b(\phi(t)-b(\phi(t-h))\phi(t) \\
&\qquad \ge \frac{c_1}{h}\int_\Omega |\phi|^{m+1} - \frac{c_2}{h} - \frac{1}{h}\int_\Omega B(\phi(t-h)).
\end{aligned}
$$

Taking into account (3.9), (3.10), (3.13), (3.14) and (3.16) and since $\int_\Omega B(\phi(t-h))$ is known, η, η_2 and h are sufficiently small we have

$$
(3.17)\qquad J_{h\lambda}(\alpha)\cdot\alpha \ge C\int_\Omega|\nabla\phi|^{r+1}+\bigg(\frac{c_1}{h}-C(\eta_2)-\eta\bigg)\int_\Omega|\phi|^{m+1}-C(\varepsilon,\eta,\eta_2,h)\ge 0
$$

for α with $|\alpha| = c$ (c is some constant) such that $\|\nabla\phi\|_{L^{r+1}(\Omega)}$ is large enough. therefore $\exists\alpha_0\in\mathbb{R}^\lambda$ such that $J_{h\lambda}(\alpha_0)=0$. Thus we have proved the existence of $u_{h\lambda}(t)$ satisfying (3.5).

The next step in the proof of the lemma is to prove the following inequalities:

$$
(3.18)\qquad \operatorname*{ess\,sup}_{0<t<T}\int_\Omega |u_{h\lambda}(t)|^{m+1} + \int_0^T \|u_{h\lambda}\|^{r+1}_{W^1_{r+1}(\Omega)} \le C_\varepsilon
$$

and

$$
(3.19)\qquad \int_0^{T-h}\int_\Omega (b(u_{h\lambda}(t+h))-b(u_{h\lambda}(t))\cdot(u_{h\lambda}(t+h)-u_{h\lambda}(t))\,dt\le C_\varepsilon h.
$$

To show (3.18) we put $v = u_{h\lambda}$ into (3.5). Hence we get

$$\int_\Omega \partial_t^{-h} b(u_{h\lambda}(t))u_{h\lambda}(t) + \int_\Omega a_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda})\nabla u_{h\lambda} \tag{3.20}$$

$$\int_{\partial\Omega} g_\varepsilon(u_{h\lambda})u_{h\lambda} - \int_\Omega f_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda})u_{h\lambda} = 0.$$

Using (H3) we have

$$\frac{1}{h}\int_\Omega (b(u_{h\lambda}(t)) - b_{h\lambda}(t-h))) \cdot u_{h\lambda(t)} + c\int_\Omega |\nabla u_{h\lambda}|^{r+1} \tag{3.21}$$

$$\leq \int_\Omega |a_\varepsilon(u_{h\lambda}, 0)||\nabla u_{h\lambda}| + \int_\Omega |g_\varepsilon(u_{h\lambda})||u_{h\lambda}|$$

$$+ \int_\Omega f_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda})||u_{h\lambda}|.$$

Now, applying in (3.21) inequalities (3.10), (3.13), (3.14) with $\phi = u_{h\lambda}$, (3.15) and (H2) we obtain

$$\frac{1}{h}\int_\Omega [B(u_{h\lambda}(t)) - B(u_{h\lambda}(t-h))] + C\int_\Omega |\nabla u_{h\lambda}|^{r+1} \tag{3.22}$$

$$\leq C_* \int_\Omega B(u_{h\lambda}(t)) + C_{**}.$$

Integrating (3.22) over $(0, t)$ (where $0 < t \leq T$) we get

$$\frac{1}{h}\int_0^t \int_\Omega B(u_{h\lambda}(t)) - \frac{1}{h}\int_{-h}^{t-h} \int_\Omega B(u_{h\lambda}(t)) + C\int_0^t \|\nabla u_{h\lambda}\|^{i+1}_{L^{r+1}(\Omega)}$$

$$\leq C_* \int_0^t \int_\Omega B(u_{h\lambda}(t)) + C'_{**}, \qquad (C'_{**} = C_{**}T).$$

Hence

$$\frac{1}{h}\int_{t-h}^t \int_\Omega B(u_{h\lambda}(t)) - \frac{1}{h}\int_{-h}^0 \int_\Omega B(u_{h\lambda}(t)) + C'_1\int_0^t \|\nabla u_{h\lambda}\|^{i+1}_{L^{r+1}(\Omega)}$$

$$\leq C_* \int_0^t \int_\Omega B(u_{h\lambda}(t)) + C'_{**}.$$

Since by (3.6) and (3.7)

$$\frac{1}{h}\int_{-h}^0 \int_\Omega B(u_{h\lambda}(t)) = \int_\Omega B(u_h^0) \leq C$$

we have

$$\int_\Omega B(u_{h\lambda}(t)) + C'_1\int_0^t \|\nabla u_{h\lambda}\|^{r+1}_{L^{r+1}(\Omega)} \leq C_* \int_0^t \int_\Omega B(u_{h\lambda}(t)) + C'_{**}.$$

Therefore applying the Gronwall inequality and (H2) we obtain

$$\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u(t)|^{m+1} + \int_0^t \|\nabla u_{h\lambda}\|^{r+1}_{L^{r+1}(\Omega)} \le C. \tag{3.23}$$

Now using inequality (2.1) with $p = r$ and (3.23) we get (3.18). Moreover

$$\int_\Omega B(u_{h\lambda}(t)) \le C \qquad \text{for } 0 < t < T. \tag{3.24}$$

Now (3.18) implies that we can choose a subsequence of $(u_{h\lambda})$ still denoted by $(u_{h\lambda})$ such that

$$u_{h\lambda} \to u_\varepsilon \qquad \text{weakly in } \ L^{r+1}(0,T;W^1_{r+1}(\Omega)) \quad \text{as} \quad (h,\lambda) \to (0,\infty). \tag{3.25}$$

In order to prove (3.19) integrate (3.5) over (t_i, t_{i+1}), where $t_i - ih$, $t_{i+1} = (i+1)h$, $i = 0, \dots, l-1$, $l = \frac{t}{h}$. We obtain

$$\begin{aligned} &\frac{1}{h}\int_{t_i}^{t_{i+1}} \int_\Omega (b(u_{h\lambda}(t)) - b(u_{h\lambda}(t-h)))v \\ &+ \int_{t_i}^{t_{i+1}} \int_\Omega a_\varepsilon(u_{h\lambda}, \nabla_{h\lambda})\nabla v - \int_{t_i}^{t_{i+1}} \int_{\partial\Omega} g_\varepsilon(u_{h\lambda})v \\ &- \int_{t_i}^{t_{i+1}} \int_\Omega f_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda})v = 0 \qquad \text{for all } \ v \in V_\lambda \end{aligned} \tag{3.26}$$

Hence changing in (3.26) variable t for $t+h$ and next putting $v = u_{h\lambda}(t+h) - u_{h\lambda}(t)$ we get

$$\begin{aligned} &\int_{t_{i-1}}^{t_i} \int_\Omega (b(u_{h\lambda}(t+h) - b(u_{h\lambda}(t))(u_{h\lambda}(t+h) - u_{h\lambda}(t)) \\ &+ h\int_{t_{i-1}}^{t_i} \int_\Omega a_\varepsilon(u_{h\lambda}(t+h), \nabla u_{h\lambda}(t+h)) \cdot \nabla(u_{h\lambda}(t+h) - u_{h\lambda}(t)) \\ &- h\int_{t_{i-1}}^{t_i} \int_{\partial\Omega} g_\varepsilon(u_{h\lambda}(t+h))(u_{h\lambda}(t+h) - u_{h\lambda}(t)) \\ &- h\int_{t_{i-1}}^{t_i} \int_\Omega f_\varepsilon(u_{h\lambda}(t+h), \nabla u_{h\lambda}(t+h)) \\ &\cdot (u_{h\lambda}(t+h) - u_{h\lambda}(t)) = 0. \end{aligned} \tag{3.27}$$

Now, summing up equalities (3.27) for $i = 1, \dots, l-1$ we have

$$
\begin{aligned}
&\int_0^{T-h}\int_\Omega (b(u_{h\lambda}(t+h)) - b(u_{h\lambda}(t)))\cdot(u_{h\lambda}(t+h) - u_{h_\lambda}(t)) \\
&+ \int_0^{T-h}\int_\Omega a_\varepsilon(u_{h\lambda}(t+h), \nabla_{h\lambda}(t+h))\cdot\nabla(u_{h\lambda}(t+h) - u_{h\lambda}(t)) \\
&- h\int_0^{T-h}\int_{\partial\Omega} g_\varepsilon(u_{h\lambda}(t+h)\cdot(u_{h\lambda}(t+h) - u_{h\lambda}(t)) \\
&- h\int_0^{T-h}\int_\Omega f_\varepsilon(u_{h\lambda}(t+h), \nabla u_{h\lambda}(t+h))\cdot(u_{h\lambda}(t+h) - u_{h\lambda}(t)) = 0.
\end{aligned} \tag{3.28}
$$

Using (3.14) with $\phi = u_{h\lambda}(t+h)$ and (3.18) we get

$$
\begin{aligned}
-h\int_0^{T-h}\int_\Omega & f_\varepsilon(u_{h\lambda}(t+h), \nabla u_{h\lambda}(t+h))u_{h\lambda}(t+h) \\
&\le h\int_0^T C\|u_{h\lambda}\|^{m+1}_{L^{m+1}(\Omega)} \le C_\varepsilon h \qquad (C_\varepsilon = C_\varepsilon(T)).
\end{aligned} \tag{3.29}
$$

In the same way we obtain

$$
h\int_0^{T-h}\int_\Omega f_\varepsilon(u_{h\lambda}(t+h), \nabla u_{h\lambda}(t+h))u_{h\lambda}(t) \le C_\varepsilon h. \tag{3.30}
$$

Similarly as (3.13) and (3.10), using (3.18) we get

$$
h\int_0^{T-h}\int_{\partial\Omega} g_\varepsilon(u_{h\lambda}(t+h))(u_{h\lambda}(t+h) - u_{h\lambda}(t)) \le C_\varepsilon h \tag{3.31}
$$

and

$$
h\int_0^{T-h}\int_\Omega a_\varepsilon(u_{h\lambda}(t+h), \nabla u_{h\lambda}(t+h))\cdot\nabla(u_{h\lambda}(t+h) - u_{h\lambda}(t)) \le C_\varepsilon h. \tag{3.32}
$$

Taking into account (3.28)–(3.32) we obtain (3.19).

Now (3.19), (3.24), (3.25) Lemma 1.9 from [1] yield

$$
b(u_{h\lambda}) \to b(u_\varepsilon) \qquad \text{in } L^1(Q_T) \tag{3.33}
$$

and hence

$$
b(u_{h\lambda}) \to b(u_\varepsilon) \qquad \text{almost everywhere in } Q_T \tag{3.34}
$$

for a subsequence of $(u_{h\lambda})$ still denoted by $(u_{h\lambda})$. Moreover, by Lemma 1.9 of [1]

$$
B(u_{h\lambda}) \to B(u_\varepsilon) \qquad \text{almost everywhere in } Q_T. \tag{3.35}
$$

Since b is strictly monotone we have

$$
u_{h\lambda} \to u_\varepsilon \qquad \text{almost everywhere in } Q_T. \tag{3.36}
$$

From Lemma 2 of [8], (3.36) and (3.18) it follows

$$u_{h\lambda} \to u_\varepsilon \qquad \text{strongly in } L^{q+1}(Q_T) \text{ for any } 0 \le q < p^* \tag{3.37}$$

and by Lemma 3 of [8]

$$u_{h\lambda} \to u_\varepsilon \qquad \text{strongly in } L^{\beta+1}(S_T) \tag{3.38}$$

for any $0 \le \beta < \frac{r(N+\min\{\beta,m\}+1)}{N}$.

Using (3.33) we can prove in the same way as in [2] that

$$\partial_t^{-h} b(u_{h\lambda}) \to \partial_t b(u_\varepsilon) \qquad \text{weakly in } L^{(r+1)/r}(0,T;V^*) \tag{3.39}$$

and that u_ε satisfies condition (i) of Definition 1.

Thus, to complete the proof of the lemma it remains only to prove strong convergence of $\nabla u_{h\lambda}$ to ∇u_ε. To do this put into (3.5) $v = u_{h\lambda} - w_{h\lambda}$, where $w_{h\lambda} \in L^{r+1}(0,T;V_\lambda)$, are approximations of u_ε in $L^{r+1}(0,T;W^1_{r+1}(\Omega)) \cap L^{r+1}(0,T;L^{m+1}(\Omega))$, i.e.

$$w_{h\lambda} \to u_\varepsilon \qquad \text{strongly in } L^{r+1}(0,T;W^1_{r+1}(\Omega)) \cap L^{r+1}(0,T;L^{m+1}(\Omega)). \tag{3.40}$$

By (H3) we have

$$\begin{aligned} \int_0^t \langle \partial_t^{-h} b(u_{h\lambda}), v\rangle + c\int_0^t \int_\Omega |\nabla v|^{r+1} \le &- \int_0^t \int_\Omega a_\varepsilon(u_{h\lambda}, \nabla w_{h\lambda})\nabla v \\ &+ \int_0^t \int_{\partial\Omega} g_\varepsilon(u_{h\lambda}) v + \int_0^t \int_\Omega f_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda}) v. \end{aligned} \tag{3.41}$$

First consider $\int_0^t \int_\Omega f_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda})v$. From (H5) and the Holder inequality it follows

$$\begin{aligned} \int_0^t \int_\Omega |f_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda})||v| &\le c\bigg(1 + \Big(\frac{1}{\varepsilon}\Big)^p + \Big(\frac{1}{\varepsilon}\Big)^s\bigg) \int_0^t \int_\Omega |u_{h\lambda} - w_{h\lambda}| \\ &\le C\bigg(\int_0^t \int_\Omega |u_{h\lambda} - w_{h\lambda}|^{r+1}\bigg)^{1/(r+1)} \end{aligned}$$

Since $r < p^*$ by (3.37) and (3.40) we obtain

$$\int_0^t \int_\Omega |f_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda})||v| \le o(1), \tag{3.42}$$

where $o(1)$ denotes any term converging to zero as $(h,\lambda) \to (0,\infty)$.

Next, by (H6), the Young inequality and (3.38) we have

$$\int_0^t \int_{\partial\Omega} |g_\varepsilon(u_{h\lambda})||v| \le C\bigg(\int_0^t \int_{\partial\Omega} |u_{h\lambda} - w_{h\lambda}|^{\alpha+1}\bigg)^{1/(\alpha+1)} = o(1). \tag{3.43}$$

Now consider

$$
\begin{aligned}
&\left|\int_0^t \int_\Omega a_\varepsilon(u_{h\lambda}, \nabla w_{h\lambda}) \cdot \nabla v\right| \\
&\le \left|\int_0^t \int_\Omega [a_\varepsilon(u_{h\lambda}, \nabla w_{h\lambda}) - a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)] \cdot \nabla v\right| \\
&\quad + \left|\int_0^t \int_\Omega a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla v\right| \qquad (3.44)\\
&\le \eta \int_0^t \int_\Omega |\nabla v|^{r+1} + C(\eta) \int_0^t \int_\Omega |a_\varepsilon(u_{h\lambda}, \nabla w_{h\lambda}) - a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)|^{(r+1)/r} \\
&\quad + \left|\int_0^t \int_\Omega a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \nabla v\right|.
\end{aligned}
$$

Since operator $A_\varepsilon \phi := a_\varepsilon(x, t, \phi)$ (where $\phi = (\phi_1, \nabla\phi_2)$) maps $L^{r+1}(Q_T)$ into $L^{(r+1)/r}(Q_T)$, it is continuous (see for example [6], pp. 20–21). Hence (3.37) (because $r < p^*$) and (3.40) yield

$$
\int_0^t \int_\Omega |a_\varepsilon(u_{h\lambda}, \nabla w_{h\lambda}) - a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)|^{(r+1)/r} \to 0 \quad \text{as } (h, \lambda) \to (0, \infty). \qquad (3.45)
$$

Moreover, since $a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \in L^{(r+1)/r}(Q_T)$ from (3.25) and (3.40) it follows

$$
\left|\int_0^t \int_\Omega a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla v\right| \to 0 \qquad \text{as } (h, \lambda) \to (0, \infty). \qquad (3.46)
$$

At last, it can be proved in the same way as in [2] that

$$
\int_0^t \langle \partial_t^{-h} b(u_{h\lambda}), v\rangle \ge \frac{1}{h} \int_{t-h}^t \int_\Omega B(u_{h\lambda}(t)) - \int_\Omega B(u_\varepsilon(t)) + o(1). \qquad (3.47)
$$

Taking into account (3.41)–(3.47) we obtain

$$
\int_\Omega (B(u_{h\lambda}(t)) - B(u_\varepsilon(t)) + C \int_0^t \int_\Omega |u_{h\lambda} - \nabla u_\varepsilon|^{r+1} \le o(1), \qquad (3.48)
$$

if η is sufficiently small.

By (3.35) and Fatou lemma

$$
\liminf_{\substack{h\to 0\\ \lambda\to\infty}} \int_\Omega (B(u_{h\lambda}(t)) - B(u_\varepsilon(t))) \ge 0.
$$

Therefore from (3.48) it follows

$$
\nabla u_{h\lambda} \to \nabla u_\varepsilon \qquad \text{strongly in } L^{r+1}((0,t) \times \Omega) \text{ for } t < T. \qquad (3.49)
$$

Hence (3.37) and (3.49) yield

$$
a_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda}) \to a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \qquad (3.50)
$$

and

$$f_\varepsilon(u_{h\lambda}, \nabla u_{h\lambda}) \to f_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \tag{3.51}$$

almost everywhere in Q_T and hence weakly in $L^{(r+1)/r}((0,t)\times\Omega)$.

Moreover, by (3.38) and Theorem 1 of [6] (see pp.20–21) we have

$$g_\varepsilon(u_{h\lambda}v \to g_\varepsilon(u_\varepsilon)v \tag{3.52}$$

in $L^1(S_t)$ for any $v \in L^{r+1}(0,T;V)\cap L^\infty(0,T;L^{m+1}(\Omega))$.

From (3.39), (3.51)–(3.52) and (3.5) it follows that u_ε satisfies (ii) of Definition 1. This completes the proof of the lemma. □

REMARK 1.When a^j, f^j, g^j and α_j depend on t, then instead of (3.5) we use the equality

$$\begin{aligned}\int_{Q_T}\partial_t^{-h}b(u_{h\lambda})v + \int_{Q_T} a_{\varepsilon h}(u_{h\lambda}, \nabla u_{h\lambda})\nabla v \\ = \int_{Q_T} f_{\varepsilon h}(u_{h\lambda}, \nabla u_{h\lambda})v + \int_{S_T} g_{\varepsilon h}(u_{h\lambda})v = 0 \qquad \forall v\in V_\lambda,\end{aligned} \tag{3.53}$$

where $a_{\varepsilon h}(z,q) = \frac{1}{h}\int_{t_{i-1}}^{t_i} a_\varepsilon(x,s,z,q)\,ds$, $f_{\varepsilon h}(z,q) = \frac{1}{h}\int_{t_{i-1}}^{t_i} f_\varepsilon(x,s,z,q)\,ds$, $g_\varepsilon(z)= \frac{1}{h}\int_{t_{i-1}}^{t_i} g_\varepsilon(x,s,z)\,ds$ for any $x\in\Omega$, $z\in\mathbb{R}^n$, $q\in\mathbb{R}^{nN}$.

3.4. Proofs of Theorems 1 and 2

PROOF OF THEOREM 1. First we prove that there exists $T^*\in(0,T]$ such that

$$\operatorname*{ess\,sup}_{0<t<T^*}\int_\Omega B(u_\varepsilon(t)) \le C. \tag{3.54}$$

and

$$\operatorname*{ess\,sup}_{0<t<T^*}\int_\Omega |u_\varepsilon(t)|^{m+1} + \int_0^{T^*} \|u_\varepsilon\|^{r+1}_{W^1_{r+1}(\Omega)} \le C. \tag{3.55}$$

In order to do this put as in [8] $v=\chi_{(0,t)}u_\varepsilon$ into the identity

$$\begin{aligned}\int_0^T \langle\partial_t b(u_\varepsilon), v\rangle + \iint_{Q_T} a_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v - \iint_{S_T} g_\varepsilon(u_\varepsilon)v \\ = \iint_{Q_T} f_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)v, \qquad \forall v\in L^{r+1}(0,T;V)\cap L^\infty(0,T;L^{m+1}(\Omega)),\end{aligned} \tag{3.56}$$

where $\chi_{(0,t)}$ is the characteristic function of $(0,t)$. Using the equality (see [2])

$$\int_0^t \langle\partial_t b(u_\varepsilon), u_\varepsilon\rangle = \int_\Omega B(u_\varepsilon(t)) - \int_\Omega B(u_0) \qquad \text{for almost all } t\in[0,T), \tag{3.57}$$

we get

$$\int_\Omega B(u_\varepsilon(t)) + \iint_{Q_t} a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot \nabla u_\varepsilon$$
$$= \int_\Omega B(u_0) + \iint_{S_t} g_\varepsilon(u_\varepsilon)u_\varepsilon + \iint_{Q_T} f_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot u_\varepsilon.$$

Applying (H3) we obtain

$$\begin{aligned}(3.58)\quad &\int_\Omega B(u_\varepsilon(t)) + c\iint_{Q_t} |\nabla u_\varepsilon|^{r+1}\\ &\le \int_\Omega B(u_0) - \iint_{Q_t} a_\varepsilon(u_\varepsilon, 0)\cdot\nabla u_\varepsilon + \iint_{S_t} g_\varepsilon(u_\varepsilon)u_\varepsilon + \iint_{Q_t} f_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon.\end{aligned}$$

Estimate the integrals in (3.58) succesively. Using (H4) and the Young inequality we have

$$\begin{aligned}(3.59)\quad \int_0^t\int_\Omega |a_\varepsilon(u_\varepsilon,0)||\nabla u_\varepsilon| \le{}& \eta\int_0^t\int_\Omega |\nabla u_\varepsilon|^{r+1} + C(\eta)\int_0^t\int_\Omega |u_\varepsilon|^{r+1}\\ &+ C(\eta)\int_0^t\int_\Omega |u_\varepsilon|^{p+1} + C(\eta).\end{aligned}$$

Next, using (H6) we get

$$(3.60)\quad \iint_{S_t} |g_\varepsilon(u_\varepsilon)||u_\varepsilon| \le C\iint_{S_t} |u_\varepsilon|^{\alpha+1} + C.$$

Applying now (H5) and the young inequality we obtain

$$\begin{aligned}(3.61)\quad \iint_{Q_t} |f_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)||u_\varepsilon| \le{}& C\iint_{Q_t} |u_\varepsilon|^{p+1} + c\eta\iint_{Q_t} |\nabla u_\varepsilon|^{r+1}\\ &+ C(\eta)\iint_{Q_t} |u_\varepsilon|^{(r+1)/(r+1-s)} + C.\end{aligned}$$

Taking into account (3.58)–(3.61) we have

$$\begin{aligned}(3.62)\quad \int_\Omega B(u_\varepsilon(t)) + C_1'\int_0^t \|u_\varepsilon\|^{r+1}_{W^1_{r+1}(\Omega)} \le{}& C_2'\Big(\iint_{Q_t} |u_\varepsilon|^{p+1} + \iint_{Q_t} |u_\varepsilon|^{r+1}\\ &+ \iint_{Q_t} |u_\varepsilon|^{(r+1)/(r+1-s)}\Big) + C_3'\iint_{Q_t} |u_\varepsilon|^{\alpha+1} + C_4'.\end{aligned}$$

Now, in view of assumptions (i) and (iii) we apply inequality (2.1) to $\int_0^t\int_\Omega |u_\varepsilon|^{p+1}$ and to $\int_0^t\int_\Omega |u_\varepsilon|^{(r+1)/(r+1-s)}$ with $p = \frac{s}{r+1-s}$, respectively. Next, by assumption (ii) we apply to $\int_0^t\int_\Omega |u_\varepsilon|^{\alpha+1}$ the interpolation inequality from [8] (see Proposition 2.) and since $r < p^*$ we use to $\int_0^t\int_\Omega |u_\varepsilon|^{r+1}$ inequality (2.1). Hence

$$\int_\Omega B(u_\varepsilon(t)) + C_1\int_0^t \|u_\varepsilon\|^{r+1}_{W^1_{r+1}(\Omega)} \le C_2\int_0^t\Big(\int_\Omega B(u_\varepsilon(s))\Big)^{\gamma+1} ds + C_3$$

for a.e. $t \in [0.T)$ and some positive constants C_1 and $\gamma \geq 0$, which are independent of ε.

Repeating further exactly the same argument as in [8] we obtain (3.54) and (3.55).

The next step relies on proving the following estimate

$$\int_0^{T^*-h}\int_\Omega (b(u_\varepsilon(t+h)-b(u_\varepsilon(t)))(u_\varepsilon(t+h)-u_\varepsilon(t))\,dt \leq Ch. \tag{3.63}$$

To do this put into (3.56) (as in [8]) $v = \chi_{(t,t+h)}w$, where $w \in V$. Then

$$\begin{aligned}\langle b(u_\varepsilon(t+h)) - b(u_\varepsilon(t)), w\rangle + \int_t^{t+h}\int_\Omega a_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla w \\ - \int_t^{t+h}\int_{\partial\Omega} g_\varepsilon(u_\varepsilon)w = \int_t^{t+h}\int_\Omega f_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)w.\end{aligned} \tag{3.64}$$

Hence for sufficiently small h we have

$$\begin{aligned}\int_\Omega (b(u_\varepsilon(t+h)) - b(u_\varepsilon(t)))w \\ \leq h\bigg(\int_\Omega |a_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\nabla w| + \int_\Omega |g_\varepsilon(u_\varepsilon)w| \\ + \int_\Omega |f_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)w| + C\bigg)\end{aligned} \tag{3.65}$$

where $C > 0$ is a constant.

Next, put $w = \zeta_\delta(u_\varepsilon(t+h) - u_\varepsilon(t)) \cdot (u_\varepsilon(t+h) - u_\varepsilon(t))$ and integrate (3.65) (with respect to t) over $(0, T^* - h)$. Then using as before (H4)–(H6), the Young inequality and the estimate

$$\|\zeta_\delta(\cdot)(u_\varepsilon(t+h) - u_\varepsilon(t)\|_V \leq \|u_\varepsilon(t+h) - u_\varepsilon(t)\|_V \qquad \text{a.e. in } (0, T^* - h)$$

we obtain

$$\begin{aligned}\int_0^{T^*-h}\int_\Omega (b(u_\varepsilon(t+h)) - b(u_\varepsilon(t)))\cdot(\zeta_\delta(\cdot)(u_\varepsilon(t+h) - u_\varepsilon(t))) \\ \leq Ch\int_0^{T^*}\bigg(\|u_\varepsilon\|^{r+1}_{W^1_{r+1}(\Omega)} + \int_\Omega |u_\varepsilon|^{p+1} + \int_\Omega |u_\varepsilon|^{(r+1)/(r+1-s)} \\ + \int_{\partial\Omega} |u_\varepsilon|^{\alpha+1} + \bigg(\int_\Omega B(u_\varepsilon)\bigg)^{\gamma+1}\bigg).\end{aligned} \tag{3.66}$$

Applying (3.54), (3.55), interpolation inequality (2.1) and Proposition 2 or Remark 2 from [8] we get that the left-hand side of (3.66) is estimated by Ch (where C is independent of ε, h, δ). Hence using the convergence

$$\zeta_\delta(\cdot)(u_\varepsilon(t+h) - u_\varepsilon(t)) \to u_\varepsilon(t+h) - u_\varepsilon(t) \qquad \text{as } \delta \to 0$$

almost everywhere on Q_{T^*-h} and Fatou lemma we obtain (3.63).

By (3.55) we can choose a subsequence of (u_ε) still denoted by (u_ε) such that

$$u_\varepsilon \to u \quad \text{weakly in } L^{r+1}(0,T^*;W^1_{r+1}(\Omega)) \text{ as } \varepsilon \to 0. \tag{3.67}$$

Thus from (3.67), (3.54), (3.63) and from Lemma 1.9 of [2] it follows that

$$b(u_\varepsilon) \to b(u) \quad \text{in } L^1(Q_{T^*}) \tag{3.68}$$

and

$$b(u_\varepsilon) \to b(u) \quad \text{a.e. in } Q_{T^*} \tag{3.69}$$

for a subsequence still denoted by (u_ε). Moreover

$$B(u_\varepsilon) \to B(u) \quad \text{a.e. in } Q_{T^*}. \tag{3.70}$$

Since b is strictly monotone we have

$$u_\varepsilon \to u \quad \text{a.e in } Q_{T^*}. \tag{3.71}$$

From Lemma 2 of [8], (3.71) and (3.55) it follows

$$u_\varepsilon \to u \quad \text{strongly in } L^{q+1}(Q_{T^*}) \text{ for any } 0 \le q \le p^* \tag{3.72}$$

and by Lemma 3 of [8]

$$u_\varepsilon \to u \quad \text{strongly in } L^{\beta+1}(S_{T^*}) \tag{3.73}$$

for any $0 \le \beta < \frac{(N+\min\{s,m\}+1)}{N}$.

Since u_ε satisfies condition (i) of Definition 1, by (3.69) we have

$$\partial_t b(u_\varepsilon) \to \partial_t b(u) \quad \text{weakly in } L^{(r+1)/r}(0,T^*;V^*) \tag{3.74}$$

and condition (i) of Definition 1 is satisfied on Q_{T^*}. As before, it remains to prove strong convergence of ∇u_ε to ∇u. We use the same argument as in the case of $\nabla u_{h\lambda}$. Thus, put into (3.56) $v = \chi_{(0,t)}(u_\varepsilon - w_\varepsilon)$, where $w_\varepsilon \in L^{r+1}(0,T^*;V) \cap L^\infty(0,T^*;L^{m+1}(\Omega))$ are approximations of u in $L^{r+1}(0,T^*;V)\cap L^\infty(0,T^*;L^{m+1}(\Omega))$, i.e.

$$w_\varepsilon \to u \quad \text{strongly in } L^{r+1}(0,T^*;V) \cap L^\infty(0,T^*;L^{m+1}(\Omega)). \tag{3.75}$$

Hence (3.75) and interpolation inequality (2.1) yield

$$w_\varepsilon \to u \quad \text{strongly in } L^{q+1}(Q_{T^*}) \text{ for any } 0 \le q < p^*. \tag{3.76}$$

Using (H3) we get

$$\int_0^t \langle \partial_t b(u), v\rangle + c\int_{Q_t} |\nabla v|^{r+1} \le -\iint_{Q_t} a_\varepsilon(u_\varepsilon, \nabla w_\varepsilon)\nabla v + \iint_{S_t} g_\varepsilon(u_\varepsilon)v + \iint_{Q_t} f_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)v. \tag{3.77}$$

By (H5) and the Holder inequality we have

$$\iint_{Q_t} |f_\varepsilon(U_\varepsilon, \nabla u_\varepsilon)||v| \le C\bigg[\iint_{Q_t} |u_\varepsilon - w_\varepsilon| + \bigg(\iint_{Q_t} |u_\varepsilon|^{p+1}\bigg)^{p/(p+1)}\bigg(\iint_{Q_t} |u_\varepsilon - w_\varepsilon|^{p+1}\bigg)^{1/(p+1)} + \bigg(\iint_{Q_T} |\nabla u_\varepsilon|^{r+1}\bigg)^{s/(r+1)}\bigg(\iint_{Q_t} |u_\varepsilon - w_\varepsilon|^{(r+1)/(r+1-s)}\bigg)^{(r+1-s)/(r+1)}\bigg].$$

Using now (3.55), (3.72), (3.76), inequality (2.1) and conditions (i) and (iii) of Theorem 1 we get

$$\iint_{Q_t} |f_\varepsilon(u_\varepsilon, \nabla u_\varepsilon||v| \le o(1), \tag{3.78}$$

where $o(1)$ denotes any term converging to zero as $\varepsilon \to 0$.

Next, by (H6) and condition (ii) of Theorem 1

$$\iint_{S_t} |g_\varepsilon(u_\varepsilon)||v| \le o(1). \tag{3.79}$$

At last

$$\begin{aligned} \bigg|\iint_{Q_t} a_\varepsilon(u_\varepsilon, \nabla w_\varepsilon)\nabla v\bigg| &\le \bigg|\iint_{Q_t} [a_\varepsilon(u_\varepsilon, \nabla w_\varepsilon) - a(u, \nabla u)]\cdot\nabla v\bigg| + \bigg|\iint_{Q_t} a(u, \nabla u)\cdot\nabla v\bigg| \\ &\le \eta\iint_{Q_t} |\nabla v|^{r+1} + C(\eta)\iint_{Q_t} |a_\varepsilon(u_\varepsilon, \nabla w_\varepsilon) - a(u, \nabla u)|^{(r+1)/r} + \bigg|\iint_{Q_t} a(u, \nabla u)\nabla v\bigg|. \end{aligned} \tag{3.80}$$

Since

$$\zeta_\varepsilon(u_\varepsilon) \to 1 \qquad \text{a.e. in } Q_{T^*}$$

by (3.72) and the Lebesgue dominated convergence theorem we have

$$\zeta_\varepsilon(u_\varepsilon)u_\varepsilon \to u \qquad \text{strongly in } L^{q+1}(Q_{T^*}) \text{ for } 0 \le q < p^*. \tag{3.81}$$

Therefore from the continuity of the operator $A\phi := a(x,t,\phi)$ (where $\phi = (\phi_1, \nabla\phi_2)$) mapping both $L^{r+1}(Q_{T^*}) \times L^{r+1}(Q_{T^*})$ into $L^{(r+1)/r}(Q_{T^*})$ for $r \geq \frac{rp}{r+1}$ and $L^{p+1}(Q_{T^*}) \times L^{r+1}(Q_{T^*})$ into $L^{(r+1)/r}(Q_{T^*})$ for $r < \frac{rp}{r+1}$ (see [6], the proof of Theorem 1, pp. 20–21) and from (3.75) it follows

$$\iint_{Q_t} |a_\varepsilon(u_\varepsilon, \nabla w_\varepsilon) - a(u, \nabla u)|^{(r+1)/r} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.82}$$

Furthermore, since $a(u, \nabla u) \in L^{(r+1)/r}(Q_{T^*})$ using (3.66) and (3.75) we have

$$\left| \iint_{Q_t} a(u, \nabla u)\nabla v \right| \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.83}$$

At last we have to consider $\int_0^t \langle \partial_t b(u_\varepsilon), v\rangle$. Since u and $u_\varepsilon$satisfy the condition (i) of Definition 1, Lemma 1.5 of [2] implies (3.57) and

$$\int_0^t \langle \partial_t b(u), u\rangle = \int_\Omega B(u(t)) - \int_\Omega B(u_0).$$

Hence by (3.74) and (3.75) we have

$$\int_0^t \langle \partial_t b(u_\varepsilon, v\rangle = \int_\Omega (B(u_\varepsilon)(t)) - B(u(t))) + o(1). \tag{3.84}$$

Therefore (for sufficiency small η) (3.77)–(3.80) and (3.82)–(3.84) yield

$$\int_\Omega (B(u_\varepsilon(t)) - B(u(t))) + C \iint_{Q_t} |\nabla u_\varepsilon - \nabla u|^{r+1} \leq o(1).$$

Hence Fatou lemma implies

$$\nabla u_\varepsilon \to \nabla u \qquad \text{strongly in } L^{r+1}((0,t) \times \Omega) \text{ for } t < T^*$$

and therefore

$$\zeta_\varepsilon(\nabla u_\varepsilon)\nabla u_\varepsilon \to \nabla u \qquad \text{strongly in } L^{r+1}((0,t) \times \Omega) \text{ for } t < T^*. \tag{3.85}$$

Using (3.81), (3.73), (3.85) and Theorem 1 from [6] (see pp. 20–21) we get

$$f_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)v \to f(u, \nabla u)v \qquad \text{in } L^1(Q_t)$$

and

$$g_\varepsilon(u_\varepsilon)v \to g(u)v \qquad \text{in } L^1(S_t) \quad \text{for any } v \in L^{r+1}(0,T^*;V) \cap L^\infty(0,T^*;L^{m+1}(\Omega)).$$

This completes the proof of the theorem.

The proof of Theorem 2 is the same as in [8].

4. Problem (1.1)–(1.3) in the case $b = \mathrm{id}$

When $b = \mathrm{id}$ system (1.1) takes the form

$$\partial_t u^j - \nabla \cdot a^j(x,t,u\nabla u) = f^j(x,t,u,\nabla u) \quad \text{in } Q_T := \Omega \times (0,T),\ j = 1,\dots,n. \tag{4.1}$$

We call a vector-valued function $u \in L^{r+1}(0,T;V) \cap (L^\infty(0,T;L^2(\Omega))$ a variational solution of (4.1) with boundary condition (1.2) and initial condition (1.3) if u satisfies Definition 1 with $b = \mathrm{id}$ and $m = 1$.

For problem (4.1), (1.2), (1.3) we obtain the following theorems analogous to Theorems 1 and 2, respectively.

THEOREM 3. *Let conditions* (H3)–(H6) *of Section* 3 *be satisfied. Moreover, let* $u_0 \in W^1_{r+1}(\Omega)$ *and* $u_0^2 \in L^1(\Omega)$. *Then there exists* $T^* \in (0,T]$ *such that problem* (4.1), (1.2), (1.3) *has a variational solution* u *on* Q_{T^*} *provided the following conditions are satisfied:*

(i) $0 < p < p^* := \max\{1, \frac{r(N+2)+2}{N}\}$;

(ii) $0 < \alpha < \frac{r(N+\min\{\alpha,1\}+1}{N}$;

(iii) $0 < s < s^* := \max\{\frac{r+1}{2}, \frac{r(N+2)+2}{N+2}\}$.

THEOREM 4. *Let conditions* (H3)–(H6) *of Section* 3 *be satisfied. Moreover, let* $u_0 \in W^1_{r+1}(\Omega)$ *and* $u_0^2 \in L^1(\Omega)$. *The problem* (4.1), (1.2), (1.3) *has a variational solution on* Q_T *for any* $T > 0$ *provided the following conditions are satisfied:*

(i) $p \le 1$ $\quad (p < 1$ *if* $p^* = 1)$;

(ii) *either* $0 < \alpha < \min\{1,r\}$ *or* $0 < r < \alpha < \frac{r(N+\alpha+1)}{N}$ *and*

$$\alpha < \begin{cases} \frac{2r}{r+1} & \text{in the case } N = 1,\\ 1 & \text{for } N = r+1,\\ \frac{N(r+1)-r+1-(N+1)^2(r-1)^2+8r(r+1)}{2(N-r-1)} & \text{otherwise.} \end{cases}$$

(iii) $s \le \frac{r+1}{2}$ $\left(s < \frac{r+1}{2} \text{ if } s^* = \frac{r+1}{2}\right)$.

The proofs of Theorems 3 and 4 are analogous to the proofs of Theorems 1 and 2.

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