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BIFURCATION OF STEADY STATES IN A MODIFIED BELOUSOV–ZHABOTINSKIĬ REACTION

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(*Submitted by K. Gęba*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

In [9] and [10] Kuhnert et al. introduced a new model for the Belousov-Zhabotinskiĭ reaction in order to take into account oxygen sensitivity and photosensitivity of the BZR. Various experiments established that especially BZ solutions in uncovered Petri dishes are very sensitive to saturation by oxygen (cf. [4]). In the first approximation, the rate of production of bromide initiated by oxygen may be expressed by a constant flow φ_1. Inhibiting effects were also observed by irradiation of BZ solutions with ultraviolet or visible light. This is included as an additional term φ_2. This approach leads, after scaling, to the following nonlinear evolution system:

$$
\begin{aligned}
\varepsilon_1 \frac{\partial u_1}{\partial t} &= d_1 \triangle_x u_1 + q u_2 - u_1 u_2 + u_1(1-u_1), \\
\varepsilon_2 \frac{\partial u_2}{\partial t} &= d_2 \triangle_x u_2 - q u_2 - u_1 u_2 + 2h u_3 + \varphi_1 + \varphi_2, \\
\frac{\partial u_3}{\partial t} &= d_3 \triangle_x u_3 + u_1 - u_3 + 3\varphi_2.
\end{aligned}
\tag{1.1}
$$

Here, $0 \le x \le l$ and we take Neumann boundary conditions. The parameters ε_i, d_i, q, h, φ_i are assumed to be non-negative and one is interested in non-negative solutions only. This system without diffusion, i.e. $d_i = 0$, is called the modified complete Oregonator. It is analyzed in [8]. Krug et al. proved the existence of solutions of system (1.1) which are periodic in time but constant in x. They established, among other properties, the stability of these periodic solutions and found conditions for supercritical and subcritical Hopf bifurcations. In the case of subcritical Hopf bifurcations only stable relaxation oscillations occur, whereas after supercritical Hopf bifurcations, stable small amplitude oscillations branch off which may undergo a transition to large amplitude relaxation oscillations. As a consequence of a theorem of Kopell and Howard [7], there exist travelling waves with sufficiently high speed. Numerical investigations showed that their amplitudes are so small that it is not sensible to seek for them in real BZ solutions. But impulse-like solutions (excitable by outer perturbation) are observed, both numerically and in the experiments. In this paper, we shall ensure the existence of "standing waves", which seems to be a new spatial structure for system (1.1).

We are interested in the steady states of (1.1). These are solutions of the system

$$\begin{aligned} 0 &= d_1 \triangle u_1 + q u_2 - u_1 u_2 + u_1(1 - u_1), \\ 0 &= d_2 \triangle u_2 - q u_2 - u_1 u_2 + 2h u_3 + \varphi_1 + \varphi_2, \\ 0 &= d_3 \triangle u_3 + u_1 - u_3 + 3\varphi_2 \end{aligned} \tag{1.2}$$

for $x \in [0, l]$ and with boundary conditions

$$\frac{\partial u_i}{\partial x}(0) = 0 = \frac{\partial u_i}{\partial x}(l) \qquad \text{for } i = 1, 2, 3. \tag{1.3}$$

The diffusion coefficients $d_i > 0$ and the length $l > 0$ are assumed to be fixed so that we are left with a four-dimensional parameter space

$$P = \{(q, h, \varphi_1, \varphi_2) : 0 < q < 1,\ h, \varphi_1, \varphi_2 \ge 0\}.$$

The restriction $0 < q < 1$ does not infer experimental limitations because q depends only weakly on the topical BZ recipe and varies between 10^{-5} and 10^{-2}, uncertainties of reaction constants included. In fact, $q < 1$ is a necessary condition for the existence of both autocatalyse and inhibition.

In [6], the set of positive homogeneous steady states, i.e. $\triangle u_i = 0$, has been studied. This is a four-dimensional manifold M in $P \times \mathbb{R}^3$. We are interested in non-homogeneous steady states bifurcating from M. Set

$$X := \left\{ u \in H^2([0, l], \mathbb{R}^3) : \frac{\partial u_i}{\partial x}(0) = 0 = \frac{\partial u_i}{\partial x}(l),\ i = 1, 2, 3 \right\}$$

and $Y := L^2([0, l], \mathbb{R}^3)$. Next, we define operators A, $C_p : X \to Y$ for any $p = (q, h, \varphi_1, \varphi_2)$ by $Au := D\triangle u$, where D is the 3×3 diagonal diffusion matrix with entries d_1, d_2, d_3 and

$$C_p(u) := \begin{pmatrix} qu_2 - u_1u_2 + u_1(1-u_1) \\ -qu_2 - u_1u_2 + 2hu_3 + \varphi_1 + \varphi_2 \\ u_1 - u_3 + 3\varphi_2 \end{pmatrix}.$$

A is a Fredholm operator of index 0 and C is completely continuous, i.e. it maps bounded sets into relatively compact ones by the Rellich imbedding theorem. Solutions of (1.2), (1.3) correspond to zeros of

$$f : P \times X \to Y, \quad f(p,u) = f_p(u) = Au + C_p(u).$$

Since the nonlinear part of (1.2) is C^∞, any solution $u \in X$ of $f_p(u) = 0$ is also C^∞.

The manifold M of homogeneous steady states can be considered as a subset of $f^{-1}(0)$. We collect some properties of M, most of which can be found in [6] and in the next Section 2. In Section 3, we study the set of singular points on M. This set consists of those points $(p,u) \in M$ where $Df_p(u)$ is not an isomorphism. If $(p,u) \in M$ is not singular, then there exist no non-homogeneous steady states near (p,u), hence (p,u) is not a bifurcation point. The set of singular points can be written as a countable union of three-dimensional submanifolds S_n of M, $n \in \mathbb{N}$. S_n consists of all $(p,u) \in M$ where $e \cdot \cos(n\pi x/l)$ lies in $\ker Df_p(u)$ for some $e \in \mathbb{R}^3 - \{0\}$. This union is not disjoint. We found numerically integers $n \neq m$ and intersection points of S_n and S_m. A detailed analysis of the singular set seems to be complicated. In Section 4 we apply the bifurcation result of Crandall and Rabinowitz [2]. Let $(p,u) \in M$ be a singular point and consider a path $\omega(t)$ on M intersecting S_n only in $\omega(0) = (p,u)$. If (p,u) lies in exactly one S_n and if a certain transversality condition holds, the result of [2] is applicable and yields an analytic family (p_s, u_s), $s \in (-\varepsilon, \varepsilon)$ of non-homogeneous steady states bifurcating from (p,u). We shall show that $p_s = p_{-s}$ for all $s \in (-\varepsilon, \varepsilon)$, which is due to an interesting sequence of symmetries of problem (1.2), (1.3). This implies, in particular, that we have a pitchfork bifurcation. In addition, the steady states u_s which we obtain satisfy a symmetry condition. Namely, if $(p,u) \in S_n$ and n is even, then $u_s(x) = u_s(l-x)$ for all $x \in [0,l]$. Moreover, setting $n = 2^\nu n'$ with $\nu \geq 1$, then $u_s(x) = u_s(x + l/2^{\nu-1})$. The result of Crandall and Rabinowitz can be applied for all singular points (p,u) in S_n except for a lower dimensional subset of S_n. As a consequence, every singular point is in fact a bifurcation point. After these local results, we prove a global bifurcation theorem in Section 5. Of course, we could apply the global one-parameter bifurcation results of Rabinowitz [12] or a generalization due to Magnus [11]. But these do not tell us how the various one-dimensional bifurcating branches we obtain in this way fit together. Instead, we shall prove that a four-dimensional set $Z_n \subset f^{-1}(0) - M$ bifurcates from S_n and satisfies a generalization of Rabinowitz' global alternative. This will be a consequence of the global implicit function theorem of [1]. If at the point $(p,u) \in S_n$ the transversality assumption of Crandall and Rabinowitz is satisfied, one may apply the usual (local) implicit function theorem to describe Z_n near (p,u). But our dimension result also holds far away from S_n, although we only use the local assumptions of Crandall and Rabinowitz. This is important, since these assumptions can be checked for many other systems different from (1.2), (1.3). In addition, we show that all steady states contained in Z_n are positive. So they are chemically meaningful. As a consequence of our global theorem, we obtain the existence of non-homogeneous positive steady states of (1.1) for certain parameter values. More precisely, we shall prove the following: The set $\pi(\overline{Z}_n \cap M)$ which contains $\pi(S_n)$ and is a subset of $\pi(\bigcup_m S_m)$ separates P in (at least) two components; here $\pi : P \times X \to P$ is the projection. There exists (at least) one such component K covered by Z_n. This means that for any parameter value $p \in K$, there exists a non-homogeneous positive steady state of (1.1). We also have a result which helps to localize K.

Let us finally make a remark concerning the stability of the solutions we obtain. Using the diffusion coefficients of [6], the bifurcating solutions are unstable near the set S of bifurcation points. As explained above, we shall show that one can continue the bifurcating steady states in a global way. In order to find stable steady states one has to analyse the sets Z_n further.

We thank Reiner Lauterbach and Björn Sandstede for discussions leading to Theorem (4.1).

2. The manifold of homogeneous steady states

The set M of positive homogeneous steady states consists of all $(p,u) \in P \times \mathbb{R}^3$ with $p = (q, h, \varphi_1, \varphi_2)$ and $u_i > 0$ such that

$$\begin{aligned} 0 &= qu_2 - u_1u_2 + u_1(1-u_1), \\ 0 &= -qu_2 - u_1u_2 + 2hu_3 + \varphi_1 + \varphi_2, \\ 0 &= u_1 - u_3 + 3\varphi_2. \end{aligned} \tag{2.1}$$

Setting $\psi = \varphi_1 + \varphi_2 + 6h\varphi_2$, an easy computation shows that $(p,u) \in M$ if and only if

$$\begin{aligned} 0 &= u_1^3 + (2h+q-1)u_1^2 + (\psi - q - 2hq)u_1 - \psi q, \\ u_2 &= (2hu_1 + \psi)/(u_1 - q), \\ u_3 &= u_1 + 3\varphi_2. \end{aligned} \tag{2.2}$$

Since $\psi q \geq 0$, there exists at least one non-negative solution u of (2.1) for any $p \in P$ and at most three. Fixing $0 < q < 1$ and $\varphi_2 \geq 0$, the set M looks qualitatively like the cusp surface (cf. Figure 1). M is a four-dimensional manifold with boundary $\partial M = M \cap (\partial P \times \mathbb{R}^3)$, where $\partial P = \{p \in P : h\varphi_1\varphi_2 = 0\}$. In fact, M can be considered as the graph of a map $(q, h, \varphi_2, u_1) \mapsto (\varphi_1, u_2, u_3)$ with

(2.3) $$\varphi_1 = u_1(u_1^2 + (q-1)u_1 - q)/(q - u_1) - 2hu_1 - \varphi_2 - 6h\varphi_2$$

and u_2, u_3 as in (2.2).

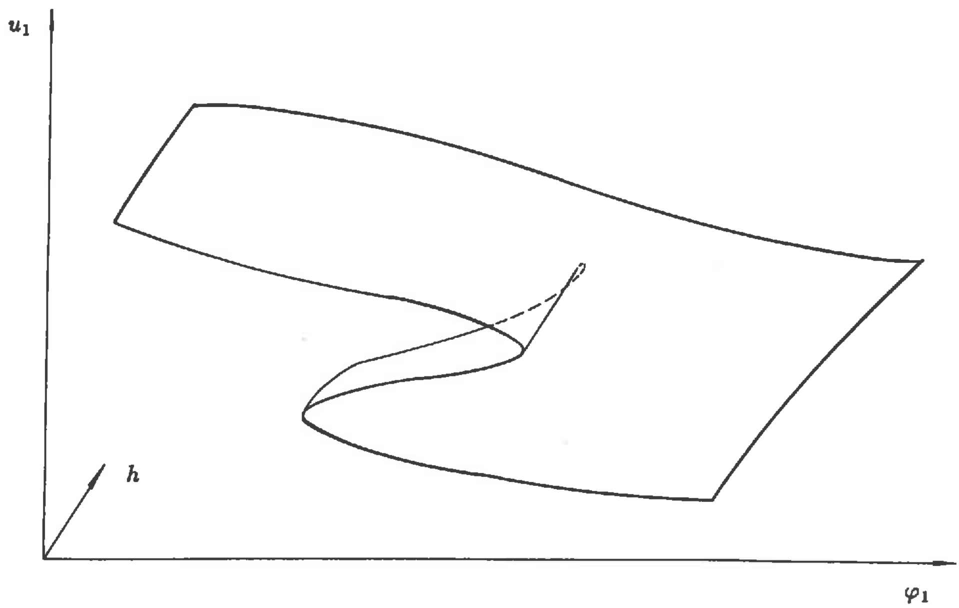

FIGURE 1. The manifold M of homogeneous steady states with some submanifolds S_n of bifurcation points. Here q and φ_2 are fixed.

LEMMA 2.4. *M is contained in the set*

$$\mathcal{D} = \{(p, u) \in P \times X \,:\, q < u_1 < 1,\ 0 < u_2 < (2h + \psi)/q,\ q + 3\varphi_2 < u_3 < 1 + 3\varphi_2\}.$$

PROOF. If $0 < u_1 \leq q < 1$, then the first equation in (2.1) implies $u_2 < 0$, a contradiction. Thus $u_1 > q$. Then, again by (2.1), $u_1(1 - u_1) > 0$, hence $u_1 < 1$. The bounds for u_2 and u_3 follow from (2.2). □

Observe that $\mathcal{D}$ is bounded away from $P \times \{0\}$, since $q > 0$. If $\varphi_1 = \varphi_2 = 0$ and q, h are arbitrary, then $u_1 = u_2 = u_3 = 0$ is a solution of (2.1). And if

$h = \varphi_1 = \varphi_2 = 0$, and q is arbitrary, then $u_1 = 1$, $u_2 = 0$, $u_3 = 1$ solves (2.1). An easy calculation shows that these are all non-negative solutions of (2.1) outside of M.

The parameter space can be divided into three disjoint subsets $P = P_1 \cup P_2 \cup P_3$ (cf. Figure 2). P_2 is the projection of the cusp "curve". It is a three-dimensional submanifold of P with $\partial P_2 = P_2 \cap \partial P$. P_2 separates P into a bounded part P_3 and an unbounded part P_1. For $p \in P_1$, respectively $p \in P_3$, there exist exactly one, respectively three, solutions of (2.1). $P_2 \cup P_3$ is contained in the set $\{(q, h, \varphi_1, \varphi_2) \in P : 0 < q \leq q^*, 0 \leq h \leq \frac{1}{2}, 0 \leq \varphi_i \leq \frac{1}{4}\}$; here $q^* \approx 0.07973$. A precise description of the manifold P_2, as well as of the projection of all cusp points, can be found in [6].

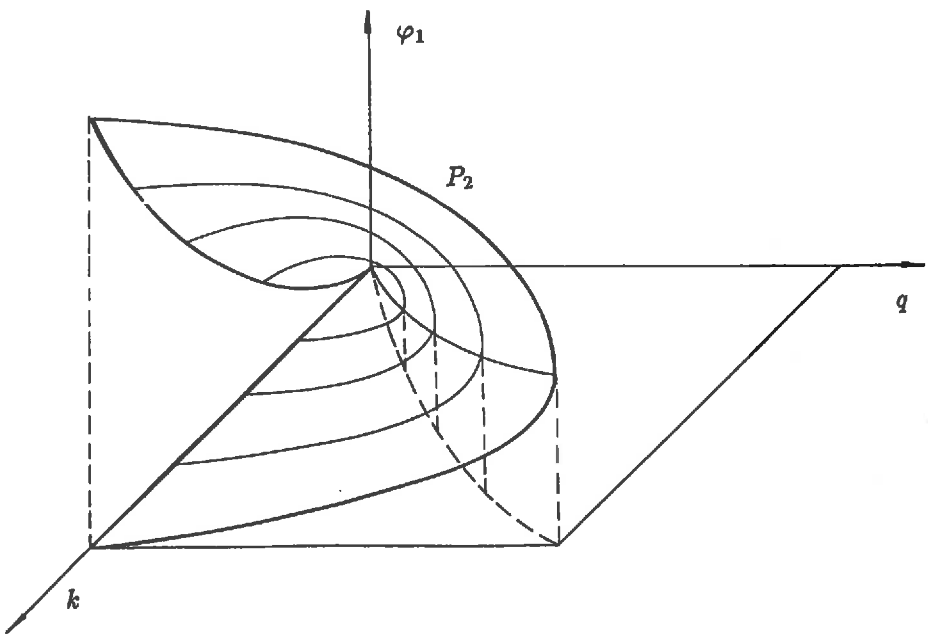

FIGURE 2. The projection P_2 of the cusp "curve" on the parameter space (φ_2 is fixed, $k{:=}h-1/2$). The curve on the surface P_2 which starts at the origin corresponds to the cusp points.

3. Singular points

As in Section 1, we consider $f : P \times X \to Y$, $f(p, u) = f_p(u) = Au + C_p(u)$. $f|M = 0$ and we want to find the set of bifurcation points of zeros of f, i.e. the set of all $(p, u) \in M$ such that there exists a sequence $(p_n, u_n) \in f^{-1}(0) - M$ with

$(p_n, u_n) \to (p, u)$ as $n \to \infty$. By the implicit function theorem, for (p, u) to be a bifurcation point, it is necessary (but not sufficient) that the linearization $Df_p(u)$ is not an isomorphism. Now $v \in X$ lies in the kernel of $Df_p(u)$ if and only if it satisfies the equation

$$(3.1) \qquad Dv'' = Lv, \qquad \text{where } L = \begin{pmatrix} 2u_1 + u_2 - 1 & u_1 - q & 0 \\ u_2 & u_1 + q & -2h \\ -1 & 0 & 1 \end{pmatrix}$$

and D is the diagonal diffusion matrix, as in the introduction. Due to the Neumann boundary condition inherent to the definition of X, $\ker Df_p(u)$ is at most three dimensional. It is spanned by elements of the form $x \mapsto e\cos(n\pi x/l)$ for some $e \in \mathbb{R}^3 - \{0\}$ and $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, let S_n be the set of all $(p, u) \in M$ such that $\ker Df_p(u)$ contains an element of the form $e\cos(n\pi x/l)$ for some $e \in \mathbb{R}^3 - \{0\}$. The elements of $S = \bigcup_{n\in\mathbb{N}} S_n$ are called singular points. These are the possible bifurcation points. In fact, we shall show later that every singular point is a bifurcation point. We first study the sets S_n.

PROPOSITION 3.2. (a) *S_n is a submanifold of M of codimension at least 1, hence* $\dim S_n \leq 3$.

(b) *If $(p, u) \in S_{n_1} \cap \ldots \cap S_{n_r}$ then $r \leq 3$. There exists a neighborhood N of (p, u) in M such that $N \cap S \subset S_{n_1} \cup \ldots \cup S_{n_r}$.*

(c) *If $n \neq m$, then $S_n \cap S_m$ is an algebraic variety of dimension at most two.*

PROOF. Fix an integer $n \geq 1$ and a point $(p, u) \in M$. Then $(p, u) \in S_n$ if and only if there exists $e \in \mathbb{R}^3 - \{0\}$ such that the function $x \mapsto e\cos(n\pi x/l)$ solves $v'' = D^{-1}Lv$. This is equivalent to the statement that $-n^2\pi^2/l^2$ is an eigenvalue of $D^{-1}L$. Using the formulas (2.2) and (2.3), we may replace u_2 (which appears on two positions in L) by q, h, φ_2 and u_1. Now we observe that h appears on three positions of L, and always linearly. This implies that h appears linearly in the constant term of the characteristic polynomial $\det(D^{-1}L - (\lambda - n^2\pi^2/l^2)\mathrm{id})$. So this constant term has the form $\alpha_n h - \beta_n$ where α_n and β_n are real polynomials depending only on q, φ_2 and u_1. Obviously, $(p, u) \in S_n$ if and only if $\lambda = 0$ is a root of the above characteristic poynomial, i.e. if $\alpha_n h - \beta_n = 0$. The explicit calculation shows that $\alpha_n > 0$ as long as $0 < q < u_1 < 1$ and $\varphi_2 \geq 0$. Using the formulas (2.2) and (2.3) once more, we obtain that S_n is the graph of a map $\sigma_n = \beta_n/\alpha_n : (q, \varphi_2, u_1) \mapsto (h, \varphi_1, u_2, u_3)$. This implies (a). It is possible that $S_n = \emptyset$ or that S_n is contained in the boundary of M. (b) follows from the fact that the 3×3-matrix $D^{-1}L$ has at most three different eigenvalues. And (c) is a consequence of the observation that the rational functions β_n/α_n and β_m/α_m (which define the h-components of σ_n and σ_m) are different for $n \neq m$. □

Proposition (3.2) is generically true, that means for an open and dense subset of the possible compact parts C_p. In the proof, we had to check that we are studying a generic case. Of course, if u has more components and if the right hand side of (1.1) is more complicated, this can be very difficult. Also, a detailed analytical study of the sets S_n seems to be complicated. Choosing chemically relevant diffusion coefficients as in [6], Tables 1 and 2, the second named author studied the singular sets numerically. Singular points $(p,u) \in S_n \cap S_m$ exist, for example, also for $p \in P_3$. Remember that in this case there are three different steady states of (1.1). The intersection of S_n and S_m can occur on all branches of M. For example, if $n = 175$ and $m = 176$, S_n and S_m intersect on the upper branch. If $n = 10$ and $m = 460$, S_n and S_m intersect on the middle branch of M.

4. Local bifurcation results

In this section, we shall prove that any singular point is in fact a bifurcation point and that except for a lower dimensional subset of the singular set, we always have a pitchfork bifurcation.

Let $\omega = (p,u) : I = (-1,1) \to M \subset P \times X$ be a differentiable path in M such that $\omega(0) = (p(0), u(0)) = (p_0, u_0)$ is singular. We also assume for simplicity that $p(\lambda) \in P - P_2$ for all $\lambda \in (-1,1)$. Remember that P_2 consists of the parameters corresponding to the cusp curve. In [2], Crandall and Rabinowitz assumed the following:

(H1) $\dim \ker Df_{p_0}(u_0) = \operatorname{codim} \operatorname{ran} Df_{p_0}(u_0) = 1$;

(H2) $\frac{\partial}{\partial \lambda} Df_{p(\lambda)}\big(u(\lambda)\big)e\big|_{\lambda=0} \notin \operatorname{ran} Df_{p_0}(u_0)$ where $e \in X$ spans $\ker Df_{p_0}(u_0)$.

(H1) implies that (p_0, u_0) lies in exactly one S_n and (H2) implies that ω intersects the singular set in (p_0, u_0) transversally.

THEOREM. (Crandall and Rabinowitz). *If* (H1) *and* (H2) *hold, then there exist differentiable functions* $\lambda : (-\varepsilon, \varepsilon) \to I$, $s \mapsto \lambda_s$, *and* $v : (-\varepsilon, \varepsilon) \to X$, $s \mapsto v_s$, *such that:*

(i) $\lambda_0 = 0$, $v_0 = 0$, $v'(0) = e$, $v_s - se \in \langle e \rangle^\perp$.

(ii) $f\big(p(\lambda_s), u(\lambda_s) + v_s\big) = 0$ *for all* $s \in (-\varepsilon, \varepsilon)$.

(iii) *There exists a neighborhood* N *of* (p_0, u_0) *in* $p(I) \times X$ *such that any zero of* f *(solution of* (1.2) *and* (1.3)*) contained in* N *either lies in* M *or is of the form* $(p(\lambda_s), u(\lambda_s) + v_s)$.

If the path ω *is analytic, then* λ *and* v *are analytic.*

For a proof, we refer the reader to [2], Theorem 1, or to [3], Theorem 28.6 and Corollary 28.1. We shall prove the following:

THEOREM 4.1. *Suppose* ω *is analytic and* (H1) *and* (H2) *hold. Let* λ *and* v *be the maps as in the above theorem. Then* $\lambda_{-s} = \lambda_s$ *for all* $s \in (-\varepsilon, \varepsilon)$. *In particular, for any parameter* $p(\lambda_s)$, $s \in (-\varepsilon, \varepsilon)$, *there exist two different nonhomogeneous steady states* $u(\lambda_s)+v_s, u(\lambda_s)+v_{-s}$ *of system* (1.1). *These steady states converge to the homogeneous steady state* u_0 *of system* (1.1) *for the parameter* p_0 *as* s *converges towards* 0.

Thus we have a pitchfork bifurcation at (p_0, u_0) if (H1) and (H2) hold. (Since M is analytic, we may always choose an analytic path ω.) Moreover, in the course of the proof, it is shown that for $(p_0, u_0) \in S_n$

$$v_{-s}(x) = \begin{cases} v_s(l-x) & \text{if } n \text{ is odd,} \\ v_s(x + l/2^\nu) & \text{if } n = 2^\nu n' \text{ with } \nu \geq 1 \text{ and } n' \text{ odd.} \end{cases}$$

In addition, if $n = 2^\nu n'$ with $\nu \geq 1$, then the steady state itself has the symmetry

$$v_s(x) = v_s(l-x) = v_s(x + l/2^{\nu-1}).$$

Now the question arises whether the assumptions of Theorem (4.1) hold in our situation. Let $T \subset S$ denote the set of all singular points such that no path ω exists which satisfies (H1) and (H2). Singular points in $S - T$ are bifurcation points by our above result.

PROPOSITION 4.2. *T is a subset of S of codimension* 1.

Since the set of bifurcation points is a closed subset of M, and since $S - T$ is dense in M by Proposition (4.2) we obtain the

COROLLARY 4.3. *Every singular point is a bifurcation point.*

Before proving these results, let us remark that the coefficients of the Taylor series of λ and v can be computed explicitely by successively differentiating the equation $f(p(\lambda_s), u(\lambda_s) + v_s) = 0$ at $s = 0$.

REMARK 4.4. Set $g : (-1,1) \times X \to Y$, $g(\lambda, u) := f(p(\lambda), u(\lambda) + u)$. Then $g(\lambda, 0) = 0$ for all λ and $g(\lambda_s, v_s) = 0$ for all s with λ_s, v_s, as in the Crandall-Rabinowitz theorem. Theorem (4.1) implies that the odd derivatives $\lambda'(0)$, $\lambda'''(0)$ etc. are 0. Using this, a straightforward calculation shows ($v'(0) = e = (e_1, e_2, e_3)$)

$$0 = \frac{\partial^2}{\partial s^2} g(\lambda_s, v_s)\Big|_{s=0} = \frac{\partial^2}{\partial u^2} g(0,0)(e,e) + \frac{\partial}{\partial u} g(0,0) v''(0).$$

Since $(\partial/\partial u)g(0,0)$ induces an isomorphism $\langle e\rangle^\perp \to \langle e^*\rangle^\perp$ and since $v''(0) \in \langle e\rangle^\perp$, we can compute

$$\begin{aligned} v''(0) &= -\Big(\frac{\partial}{\partial u} g(0,0)\Big)^{-1}\Big(\frac{\partial^2}{\partial u^2} g(0,0)(e,e)\Big) \\ &= \Big(A + \frac{\partial}{\partial u} C(p_0,u_0)\Big)^{-1}\begin{pmatrix} 2e_1^2 + 2e_1e_2 \\ 2e_1e_2 \\ 0 \end{pmatrix}. \end{aligned}$$

Differentiating once more and taking the L^2-scalar product with e^*, one obtains

$$\begin{aligned} 0 &= \Big\langle \frac{\partial^3}{\partial s^3} g(\lambda_s, v_s)\Big|_{s=0}, e^*\Big\rangle \\ &= 3\lambda''(0)\Big\langle \frac{\partial^2}{\partial\lambda\partial u} g(0,0)e, e^*\Big\rangle + 3\Big\langle \frac{\partial^2}{\partial u^2} g(0,0)(v''(0),e), e^*\Big\rangle. \end{aligned}$$

Here we also used that $(\partial^3/\partial u^3)g(\lambda,u) = 0$, due to the special form of the compact part C of f. Observe that the coefficient of $\lambda''(0)$ on the right hand side is not 0 by assumption (H2). Therefore we get

$$\begin{aligned} \lambda''(0) &= -\frac{\langle \frac{\partial^2}{\partial u^2} g(0,0)(e, v''(0), e^*\rangle}{\langle \frac{\partial^2}{\partial\lambda\partial u} g(0,0)e, e^*\rangle} \\ &= -\frac{\langle \frac{\partial^2}{\partial u^2} C(p_0,u_0)(e, v''(0)), e^*\rangle}{\langle \frac{\partial^2}{\partial p\partial u} C(p_0,u_0)(p'(0), e) + \frac{\partial^2}{\partial u^2} C(p_0,u_0)(e, u'(0)), e^*\rangle}. \end{aligned}$$

Explicitly, setting $p'(0) = \overline{p} = (\overline{q}, \overline{h}, \overline{\varphi}_1, \overline{\varphi}_2)$ and $w = (w_1, w_2, w_3) \in X$, we have

$$\frac{\partial^2}{\partial u^2} C(p_0,u_0)(e,w) = -\begin{pmatrix} 2e_1w_1 + e_1w_2 + e_2w_1 \\ e_1w_2 + e_2w_1 \\ 0 \end{pmatrix}$$

and

$$\frac{\partial^2}{\partial p\partial u} C(p_0,u_0)(\overline{p}, e) = -\begin{pmatrix} \overline{q}e_2 \\ -\overline{q}e_2 + 2\overline{h}e_3 \\ 0 \end{pmatrix}.$$

Obviously, $\lambda''(0) \neq 0$ if and only if $\langle(\partial^2/\partial u^2)C(p_0,u_0)(e, v''(0)), e^*\rangle \neq 0$. Similar to the proof of Proposition (3.2), one can show that $\lambda''(0) \neq 0$ except for a lower dimensional subset of the singular set S. (Observe that $(\partial^2/\partial u^2)C(p_0,u_0)$ is independent of (p_0,u_0), whereas e, e^* and $v''(0)$ depend on (p_0,u_0). In fact, for $(p_0,u_0) \in S_n$, $e = \overline{e}\cdot\cos(n\pi x/l)$ and the components of $\overline{e}$ are of the form $\overline{e}_i = \alpha_i/\beta_i$ with α_i, β_i real polynomials and similarly for e^* and $v''(0)$.) If $\lambda''(0) \neq 0$, then $\lambda''(0)$ determines the direction of bifurcation which is given by $\lambda''(0)\cdot p'(0)$. Thus, if $\lambda''(0) > 0$, resp. $\lambda''(0) < 0$, there exist non-homogeneous steady states of (1.1) for the parameter values $p(\lambda)$ with $\lambda > 0$, resp. $\lambda < 0$. The second named author found numerically examples where $\lambda''(0)$ changes sign, which corresponds to a passage from subcritical to supercritical bifurcation.

We now come to the proof of Theorem (4.1). To do this, we define a sequence $X_0 = X \supset X_1 \supset X_2 \supset \dots$ of subspaces of $X \subset H^2([0,l],\mathbb{R}^3)$ and bounded linear maps $T_\nu : X_\nu \to X_\nu$ such that T_ν is an involution (i.e. $T_\nu \circ T_\nu = Id_{X_\nu}$) and $X_{\nu+1} = \operatorname{Fix} T_\nu = \{u \in X_\nu : T_\nu u = u\}$. First, we set

$$T_0 : X_0 \to X_0, \qquad T_0 u(x) := u(l-x).$$

Obviously, T_0 is a linear involution. If $T_0 u = u$, then $u(0) = u(l)$. Therefore, we may consider the elements u of $X_1 := \operatorname{Fix} T_0$ as l-periodic maps $u : \mathbb{R} \to \mathbb{R}^3$ with $u(x) = u(-x)$. Now we define for $\nu \geq 1$

$$X_{\nu+1} := \{u \in X_1 : u(l/2^\nu - x) = u(x) \text{ for all } x \in \mathbb{R}\}$$

and

$$T_\nu : X_\nu \to X_\nu, \qquad T_\nu u(x) := u(l/2^\nu - x).$$

It is clear that $X_{\nu+1} = \operatorname{Fix} T_\nu \subset X_\nu$ for any $\nu \geq 0$. To see that $T_\nu(X_\nu) \subset X_\nu$, first observe that $T_\nu u(x) = u(x + l/2^\nu)$ and $u'(l/2^\nu) = 0$ for $u \in X_\nu$. Thus, $(T_\nu u)'(0) = (T_\nu u)'(l) = 0$. Furthermore, $T_{\nu-1} \circ T_\nu = T_\nu$, which implies $T_\nu(X_\nu) = T_\nu(\operatorname{Fix} T_{\nu-1}) \subset \operatorname{Fix} T_{\nu-1} = X_\nu$. Next, let Y_ν be the L^2-closure of X_ν in $Y = Y_0 = L^2([0,l],\mathbb{R}^3)$. Since the L^2-norm of T_ν is 1, T_ν extends to a bounded linear involution on Y_ν, which we continue to denote $T_\nu : Y_\nu \to Y_\nu$. It is also easy to check that a Hilbert space base of X_ν and Y_ν is given by $e_j \cos(k2^\nu \pi x/l)$ with e_1, e_2, e_3 a base of $\mathbb{R}^3$ and $k \geq 0$. The intersection $\bigcap_{\nu\geq 0} Y_\nu = \bigcap_{\nu \geq 0} X_\nu \cong \mathbb{R}^3$ consists of all constant functions $[0,l] \to \mathbb{R}^3$.

Next, one checks that $f : P \times X \to Y$ commutes with all involutions T_ν, i. e. $f_p \circ T_\nu = T_\nu \circ f_p$ for all $p \in P$. This implies that $f(P \times X_\nu) \subset Y_\nu$ for all $\nu \geq 0$. Let e, resp. e^*, span $\ker Df_{p_0}(u_0) \subset X$, resp. $(\operatorname{ran} Df_{p_0}(u_0))^\perp \subset Y$. There exist $n \in \mathbb{N}$ and $\overline{e}$, $\overline{e}^* \in \mathbb{R}^3 - \{0\}$ such that $e = \overline{e} \cdot \cos(n\pi x/l)$ and $e^* = \overline{e}^* \cdot \cos(n\pi x/l)$. Let $\nu \geq 0$ be such that $n = 2^\nu n'$ with n' odd. Then, $e \in X_\nu - X_{\nu+1}$ and $\overline{e} \in Y_\nu - Y_{\nu+1}$. Now consider the map

$$g : (-1,1) \times X_\nu \to Y_\nu, \qquad g_\lambda(u) = g(\lambda, u) := f\big(p(\lambda), u(\lambda) + u\big).$$

By construction, g is analytic, $g(\lambda, 0) = 0$, $\ker Dg_0(0) = \langle e \rangle$, $(\operatorname{ran} Dg_0(0))^\perp = \langle e^* \rangle$ and g_λ commutes with T_ν since $T_\nu u(\lambda) = u(\lambda)$ for all λ.

Observe that $\langle e \rangle$ and $\langle e^* \rangle$ are invariant under T_ν. In fact, $T_\nu e = -e$ and $T_\nu e^* = -e^*$. Let $P : Y_\nu \to \langle e^* \rangle$ and $P^\perp : Y_\nu \to \langle e^* \rangle^\perp$ denote the orthogonal projections. The equation $g(\lambda,u)=0$ is equivalent to the pair of equations

$$(4.5)\qquad \begin{aligned} Pg(\lambda,\alpha e+v)&=0\\ P^{\perp}g(\lambda,\alpha e+v)&=0\end{aligned}$$

where $\alpha\in\mathbb{R}$ and $v\in\langle e\rangle^{\perp}\subset X_{\nu}$. We can apply the implicit function theorem to the second equation in (4.5). This yields an analytic map $h:(\lambda,\alpha e)\mapsto h(\lambda,\alpha e)\in\langle e\rangle^{\perp}$ which is defined in a neighborhood of $(0,0)$ in $\mathbb{R}\times\langle e\rangle$ and has the following property: In a neighborhood of the origin of $I\times\langle e\rangle\times\langle e\rangle^{\perp}=I\times X_{\nu}$, the second equation in (4.5) holds if and only if $v=h(\lambda,\alpha e)$. This implies that h commutes with T_{ν}. After this Lyapunov-Schmidt reduction, it remains to solve the bifurcation equation

$$(4.6)\qquad b(\lambda,\alpha e):=Pg(\lambda,\alpha e+h(\lambda,\alpha e))=0.$$

The map b satisfies the symmetry condition $b(\lambda,-\alpha e)=-b(\lambda,\alpha e)$. We can apply the Crandall-Rabinowitz theorem to equation (4.6) and obtain an analytic map $\lambda:(-\varepsilon,\varepsilon)\to I$ such that all solutions of (4.6) near the origin are either of the form $(\lambda,0)$ or of the form (λ_s,se). The map $v:(-\varepsilon,\varepsilon)\to X$ appearing in the Crandall–Rabinowitz theorem is given by $v_s=se+h(\lambda_s,se)$. The symmetry of b now implies $\lambda_{-s}=\lambda_s$, as claimed. □

Finally, we have to prove Proposition (4.2). First of all, T contains all points on the cusp curve. This set has codimension 1 in S. Now consider a singular point $(p_0,u_0)\in T$ not lying on the cusp curve. If (H1) is not satisfied, then either there exist $m\neq n$ with $(p_0,u_0)\in S_n\cap S_m$ or there exist two linearly independent eigenfunctions $e_1=\overline{e}_1\cos(n\pi x/l)$ and $e_2=\overline{e}_2\cos(n\pi x/l)$ of $Df_{p_0}(u_0)$. We already showed in Proposition (3.2c) that the set $S_n\cap S_m$ is an algebraic variety of dimension at most two, hence codimension (w.r.t. S) at least 1. If e_1 and e_2 solve $v''=D^{-1}Lv$, then the characteristic polynomial $\det\big(D^{-1}L-(\lambda-n^2\pi^2/l^2)\mathrm{id}\big)$ has $\lambda=0$ as a multiple root. Looking at the constant term of this polynomial implied that $(p_0,u_0)\in S_n$ if and only if $h=\beta_n/\alpha_n$ with α_n and β_n, as in the proof of Proposition (3.2). Now, if $\lambda=0$ is a multiple root, then also the coefficient of the linear term must be zero. This gives a different rational map for h, $h=\delta_n/\gamma_n$. Therefore, the intersection $S_n\cap S_m$ which is defined by $\delta_n/\gamma_n=\beta_n/\alpha_n$ has codimension at least 1 in S_n. Finally, (H2) is equivalent to

$$0\neq\left\langle\frac{\partial^2}{\partial p\partial u}f(p_0,u_0)(p'(0),e),e^*\right\rangle=\langle\overline{q}e_2,e_1^*\rangle+\langle 2\overline{h}e_3-\overline{q}e_2,e_2^*\rangle$$

where $p'(0)=(\overline{q},\overline{h},\overline{\varphi}_1,\overline{\varphi}_2)$, $e^*=(e_1^*,e_2^*,e_3^*)$ spans $(\ker Df_{p_0}(u_o))^{\perp}$ and $\langle -,-\rangle$ denotes the L^2-scalar product. Remember, from Remark (4.4), that e and e^* depend on (p_0,u_0). One can now argue, as in the proof of Proposition (3.2) or Remark (4.4), to show that the equation $\langle\overline{q}e_2,e_1^*\rangle+\langle 2\overline{h}e_3-\overline{q}e_2,e_2^*\rangle=0$ is only satisfied for (p_0,u_0)

in a lower dimensional subset of S_n. (We are even allowed to vary $\overline{q}$ and $\overline{h}$ so that we obtain two equations, one for $\overline{q} = 0$ and one for $\overline{h} = 0$.) □

5. Global bifurcation results

The goal of this section is to study the bifurcating set of steady states of (1.1) away from the set of bifurcation points. This also yields results about the existence of non-homogeneous steady states for certain parameter values. We first prove that the solutions we obtain this way are positive. Let Z be the connected component of M in $f^{-1}(0)$.

PROPOSITION 5.1. *Z is contained in the set*

$$\mathcal{D} = \{(p,u) \in P \times X \;:\; q < u_1 < 1,\ 0 < u_2 < (2h+\psi)/q,\ q+3\varphi_2 < u_3 < 1+3\varphi_2\}$$

with $\psi = \varphi_1 + \varphi_2 + 6h\varphi_2$.

As a consequence of this proposition, all solutions bifurcating from M are positive and they are bounded over any compact subset of the parameter space. Since f is of the form $f = A + C$ with A a linear Fredholm operator of index 0 and C a compact map, $f^{-1}(0) \cap (Q \times X)$ and $Z \cap (Q \times X)$ are compact for any compact subset Q of P. In other words, the projection $\pi : Z \subset P \times X \to P$ is a proper map.

Let $\omega = (p,u) : I = (-1,1) \to M$ be an analytic path as in Section 4 satisfying (H1) and (H2). In particular, ω intersects the singular set S in $\omega(0) = (p_0, u_0) \in S_n$ for some n. Let $n = 2^\nu n'$ with n' odd and X_ν, Y_ν be the subspaces of X, Y from Section 4. We can describe the bifurcating set of zeros of f which lies over $p(I)$ near (p_0, u_0) as the graph of a map $(-\varepsilon, \varepsilon) \ni s \mapsto \big(p(\lambda_s), u(\lambda_s) + v_s\big) \in P \times X_\nu$ where $\lambda : (-\varepsilon, \varepsilon) \to I$ and $v : (-\varepsilon, \varepsilon) \to X_\nu$ are analytic. The questions we address in this section are the following: How do these various graphs fit together when we change (p_0, u_0) and ω? Do they form a four-dimensional subset of $f^{-1}(0)$ or even of $f^{-1}(0) \cap P \times X_\nu$? What happens far away from the singular set S? The first question could be solved locally with the help of the implicit function theorem if we assume certain transversality conditions. The last question could be dealt with the help of the one-parameter global bifurcation theory of Rabinowitz [12] or some generalization (see e.g. [11]), at least as long as we avoid the critical parameter set P_2 (the cusp curve). But in order to express a global result similar to the Rabinowitz alternative and to have a result on the dimension, we use the global implicit function theorem of [1]. This also has the advantage that we just need the local knowledge on f near M as in Section 4. The only global result on $f^{-1}(0)$ important for the following discussion is the existence of the a-priori bounds for Z obtained in Proposition (5.1).

We need some preparations for formulating our global result. First, we introduce an appropriate version of local dimension, the small inductive dimension, which works in the absence of a manifold structure. Let T be a topological space. Then we say

$\dim T = -1$ if and only if $T = \emptyset$;

$\dim T \le n$ at a point $x \in T$ if only if x has arbitrarily small neighborhoods U in T with $\dim \partial U \le n - 1$;

$\dim T \le n$ if and only if $\dim T \le n$ at every point $x \in T$.

Of course, $\dim T = n$ means $\dim T \le n$ and $\dim T \not\le n-1$. If T is a manifold, then $\dim T$ is the usual dimension. This and other facts about dim can be found in the book by Hurewicz and Wallman [5].

Now assume that the analytic map $\lambda : (-\varepsilon, \varepsilon) \to I$ from above is not constant. This is true generically since $\lambda''(0) \neq 0$, if (p_0, u_0) does not belong to a subset of codimension 1 of S; cf. Remark (4.4). Fix some $s \in (0, \varepsilon)$ and write $p_1 := p(\lambda_s)$, $u_1 := u(\lambda_s) + v_s$. If $\delta > 0$ is small enough, then $f(p_1, u) \neq 0$ for all u with $0 < \|u - u_1\| \le \delta$. In particular, the sphere

$$\Sigma = \{(p, u) \in P \times X_\nu : p = p_1, \|u - u_1\| = \delta\}$$

is contained in $P \times X_\nu - f^{-1}(0)$.

THEOREM 5.2. *There exists a connected subset Z_1 of $(Z - M) \cap (P \times X_\nu)$ containing (p_1, u_1) with $\dim Z_1 \ge \dim P = 4$ at every point and such that the inclusion*

$$i_1 : \big(Z_1 \cup M, (Z_1 \cup M) \cap (\partial P \times X_\nu)\big) \hookrightarrow (P \times X_\nu - \Sigma, \partial P \times X_\nu)$$

is not nullhomotopic.

This means that the subset $Z_1 \cup M$ cannot be deformed continuously into $\partial P \times X_\nu$ inside $P \times X_\nu - \Sigma$ if the points on the boundary of $Z_1 \cup M$ remain inside $\partial P \times X_\nu$. Observe that this statement is false if we replace $Z_1 \cup M$ by M. Thus, $Z_1 \cup M$ and Σ are linked. One can also show that the dual inclusion $\Sigma \hookrightarrow P \times X_\nu - (Z_1 \cup M)$ is not compact nullhomotopic, which means that Σ cannot be deformed to a point in $P \times X_\nu - (Z_1 \cup M)$ via maps of the form $Id +$ *compact*. But this needs strong tools from algebraic topology. We illustrate Theorem (5.2) in the case $P \subset \mathbb{R}$ and $X_\nu = \mathbb{R}^2$ in Figure 3. Theorem (5.2) is also true if we replace X_ν by X both in the definition of Σ and in the statement of the theorem. In the one-parameter situation which Rabinowitz considers in [12], his global alternative is a consequence of the statement that i_1 is not nullhomotopic.

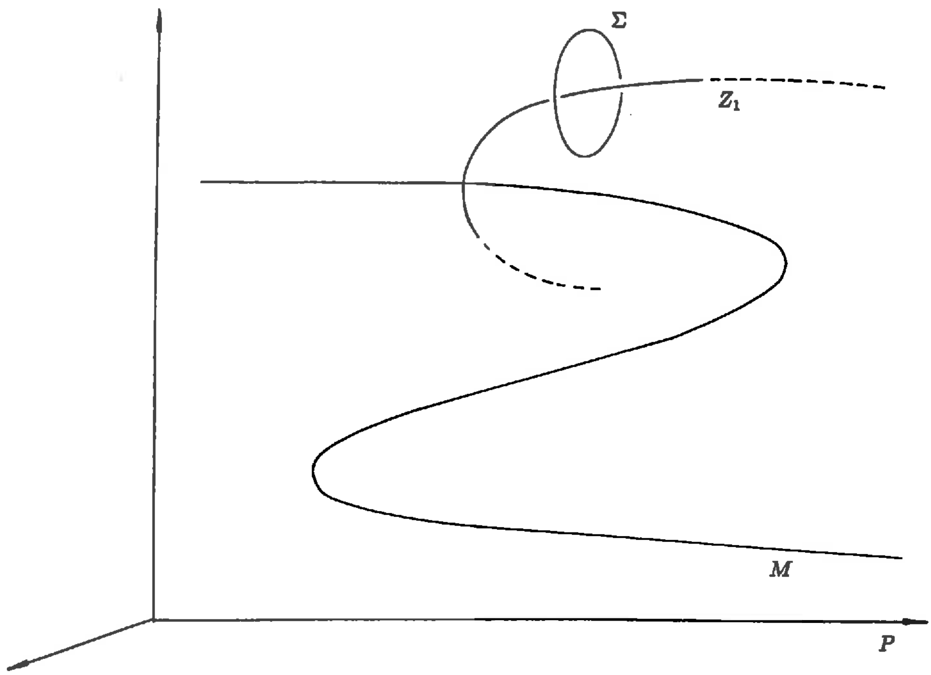

FIGURE 3. A schematic illustration of Theorem (5.2) in $(P \times X_\nu)$–space. Σ and $Z_1 \cup M$ are linked.

Next, we address the question whether one can continue the non-homogeneous steady state (p_1, u_1) to another one (p_2, u_2) with given parameter value p_2. Let Z_0 be the connected component of (p_0, u_0) in $(P \times X_\nu) \cap (Z - M) \cup \{(p_0, u_0)\}$. Z_0 is the set of zeros of f in $P \times X_\nu$ that bifurcate from (p_0, u_0). It contains the set Z_1 from Theorem (5.2). $\pi(\overline{Z}_0 \cap M) \subset P$ separates P into at least two components; here $\pi : P \times X \to P$ is the projection, as usual. Suppose that the path $p : (-1, 1) \to P$ intersects $\pi(\overline{Z}_0 \cap M)$ only in p_0. Again, this is generically true. Let K_+ be the component of $P - \pi(\overline{Z}_0 \cap M)$ containing $p(+1/2)$ and K_- the one containing $p(-1/2)$.

THEOREM 5.3. *$\pi(Z_0)$ contains K_+ or K_-.*

This means, if there exists a parameter $p_- \in K_-$ so that $p_- \notin \pi(Z_0)$, then for any $p \in K_+$, there exists $u \in X$ with $(p,u) \in Z_0$. In particular, system (1.1) has a non-homogeneous positive steady state for any parameter value $p \in K_+$. The next result helps to decide which component of $P - \pi(\overline{Z}_0 \cap M)$ is covered by Z_0.

PROPOSITION 5.4. *If $h = \varphi_1 = \varphi_2 = 0$ and $0 < q < 1$ is arbitrary, then system (1.1) has no non-homogeneous steady state contained in $\{u \in X : 0 \le u_1 \le 1,\ u_2 \ge 0,\ u_3 \ge 0\}$.*

This proposition, together with Theorem (5.3), implies the following global alternative for the set Z_0. Let $n \in \mathbb{N}$ be such that $(p_0, u_0) \in S_n$.

COROLLARY 5.5. *At least one of the following is true:*

(i) $\overline{Z}_0 \cap M \not\subset S_n$.
(ii) *There exists a component K of $P - \pi(S_n)$ such that $K \subset \pi(Z_0)$ and K contains no elements of the form $(q,0,0,0)$, $0 < q < 1$.*

If (i) is false and $P - \pi(S_n)$ has two components, then one of them must contain all the points $(q,0.0,0)$, $0 < q < 1$, and the other one is covered by Z_0. Part (i) is equivalent to saying that Z_0 connects S_n to S_m for some $m \ne n$. If $n = 2^\nu n'$ and $m = 2^\mu m'$, this is only possible if $\mu \ge \nu$ because Z_0 is contained in $P \times X_\nu$, whereas the zeros of f which bifurcate from S_m and which are close to S_m are contained in $P \times (X_\mu - X_{\mu+1})$. Moreover, assume $\mu > \nu$ and let Z_0' be the set of zeros of f which bifurcate from S_m and which are contained in $P \times X_\mu$. Then, in case (i) of Corollary (5.5), Z_0 can be considered as a secondary branch bifurcating from Z_0'. In order to find out whether (i) or (ii) hold for certain values of n and m, one needs a much more detailed study of f. In particular, the local information about f near M which we use in this paper does not suffice to yield this kind of result.

Before proving these results, let us say a few words about the calculations needed. This also indicates the range of applicability of our methods to other systems than (1.1). Theorem (5.2) is a consequence of the assumptions (H1) and (H2). Thus, local knowledge about f near (p_0, u_0) gives a global result, at least if f_p is of the form $L_p + C_p$ with L_p a linear Fredholm operator of index 0 and C_p a compact map. (In our case, $L_p = A$ is independent of $p \in P$.) The a-priori-bounds in Proposition (5.1) and the non-existence result in Proposition (5.4) require some knowledge of f away from the set M. Theorem (5.3) is a consequence of (H1) and (H2) and the a-priori-bounds (5.1).

Now we come to the proofs of our results. We start with Proposition (5.4), since it is needed in the proof of Proposition (5.1). Let $u = (u_1, u_2, u_3)$ be a steady state of (1.1) with $h = \varphi_1 = \varphi_2 = 0$ and $0 < q < 1$ arbitrary. Assume $0 \le u_1 \le 1$,

$u_2 \geq 0$ and $u_3 \geq 0$. If u_2 takes its maximum at a point x^*, we obtain

$$0 = d_2 u_2''(x^*) - q u_2(x^*) - u_1(x^*) u_2(x^*) \leq -u_2(x^*) \cdot (q + u_1(x^*)).$$

Since $q > 0$, we obtain $u_2(x^*) = 0$, hence $u_2 \equiv 0$. Then u_1 is a solution of the system

$$\begin{aligned} u_1' &= v_1, \\ v_1' &= u_1(u_1 - 1)/d_1, \\ v_1(0) &= v_1(l) = 0. \end{aligned} \tag{5.6}$$

This system has two constant solutions, $(0,0)$ and $(1,0)$. Furthermore, if $0 < u_1 < 1$, then $v_1' < 0$. Thus, any solution (u_1, v_1) of (5.6) starting at $0 < u_1(0) < 1$ must leave the range $0 \leq u_1 \leq 1$. So either $u_1 \equiv 0$ or $u_1 \equiv 1$. Since u_3 solves the equation $d_3 u_3'' + u_1 - u_3 = 0$ with $u_3'(0) = u_3'(l) = 0$, we get either $u_3 \equiv 0$ or $u_3 \equiv 1$. □

To prove Proposition (5.1), observe that $M \subset \mathcal{D}$ by Lemma (2.4). If Z is not contained in $\mathcal{D}$, then Z must intersect the boundary of $\mathcal{D}$ because Z is connected. Let (p, u) be an element of $Z \cap \partial\mathcal{D}$. This implies $f(p, u) = 0$ and $q \leq u_1 \leq 1$, $0 \leq u_2 \leq (2h + \psi)/q$, $q + 3\varphi_2 \leq u_3 \leq 1 + 3\varphi_2$ and there exists $x^* \in [0, l]$ so that at least one equality holds. Remember that $f(p, u) = 0$ forces u to be continuous (even differentiable). We shall show that equality leads to a contradiction. Suppose first that $u_1(x^*) = q$. Then $u_1''(x^*) \geq 0$ and thus $0 = d_1 u_1''(x^*) + q(1 - q) > 0$, a contradiction. Next, suppose u_2 attains its minimum in x^*; hence $u_2''(x^*) \geq 0$. Then

$$0 = d_2 u_2''(x^*) + 2h u_3(x^*) + \varphi_1 + \varphi_2$$

and, therefore, $h = \varphi_1 = \varphi_2 = 0$ since $u_3(x^*) \geq q > 0$. But then, Proposition (5.4) tells us that u must be constant. Since $u_2 > 0$, we get $(p, u) \in M \subset \mathcal{D}$ by Lemma (2.4) and the remark following it. Similarly, the reader may verify that $u_3(x^*) = q + 3\varphi_2$ implies $u_1(x^*) = q$, that $u_1(x^*) = 1$ implies $u_2(x^*) = 0$ and that $u_3(x^*) = 1 + 3\varphi_2$ implies $u_1(x^*) = 1$. Thus, all these cases lead to a contradiction. Finally, if $u_2(x^*) = (\varphi_1 + \varphi_2 + 6h\varphi_2 + 2h)/q$, then $u_2''(x^*) \leq 0$ and

$$\begin{aligned} 0 &= d_2 u_2''(x^*) - 6h\varphi_2 - 2h - u_1(x^*) \cdot u_2(x^*) + 2h u_3(x^*) \\ &\leq -u_1(x^*) \cdot u_2(x^*). \end{aligned}$$

Here we used that $u_3(x^*) \leq 1 + 3\varphi_2$. This implies $u_2(x^*) = 0$, a contradiction. □

Next we prove Theorem (5.2) by reducing it to the global implicit function theorem (2.1) of [1]. For the convenience of the reader, we state it here, adapted to our situation. Set $Q := \{p \in P : h, \varphi_1, \varphi_2 > 0\}$, $E := Q \times X_\nu$ and $g := f|E$. Q is a connected differentiable manifold without boundary. Now we add a point at infinity, $E^+ = E \cup \{\infty\}$. A neighborhood basis of ∞ consists of complements of sets of the form $K \times B$ with $K \subset Q$ compact and $B \subset X_\nu$ bounded. If $F \subset E$ is a closed and locally compact subset of E, then $F^+ = F \cup \{\infty\} \subset E^+$ is the one-point compactification of F. Consider the point $(p_1, u_1) \in E$. u_1 is an isolated zero of $g_{p_1} = f_{p_1}$ and $\delta > 0$ is so small that $f_{p_1}(u) \neq 0$ if $0 < \|u - u_1\| \leq \delta$. Choose an isomorphism $A^- : Y_\nu \to X_\nu$ such that $A^- \circ A$ is of the form $Id_{X_\nu} + compact$. Then the Leray-Schauder degree $\deg(A^- \circ g_{p_1}, u_1 + \delta \cdot BX_\nu, 0)$ is defined; BX_ν is the closed unit ball in X_ν. This degree is called the local degree of f_{p_1} at u_1, $\deg(g_{p_1}, u_1)$. The proof of the following theorem can be found in [1], (2.1).

Global implicit function theorem. *If $|\deg(g_{p_1}, u_1)| = 1$, then there exists a closed connected subset Z_0 of $g^{-1}(0)$ with $\dim Z_0 \geq 4 = \dim Q$ at every point. Furthermore, the inclusion $i_0 : Z_0^+ \hookrightarrow E^+ - \Sigma$ induces a non-zero map*

$$i_0^* : H^4(E^+ - \Sigma) \to H^4(Z_0^+).$$

Here, $\Sigma = \{(p_1, u) : \|u - u_1\| = \delta\}$. H^* is the Alexander-Spanier (or Čech) cohomology; cf. [13], Section 6.4. Actually, in [1], it is required that $Dg_{p_1}(u_1)$ is an isomorphism but only $|\deg(g_{p_1}, u_1)| = 1$ is used. In our situation, it suffices even to assume $|\deg(g_{p_1}, u_1)| \neq 0$; see the remarks following Theorem (2.1) in [1].

Let us check whether the assumption $|\deg(g_{p_1}, u_1)| = 1$ is satisfied. The hypotheses (H1) and (H2) imply that $d(\lambda) := \deg(f_{p(\lambda)}, u(\lambda))$ changes sign as λ passes 0. Since $Df_{p(\lambda)}(u(\lambda))$ is an isomorphism for $\lambda \neq 0$, we have $|d(\lambda)| = 1$. Due to the homotopy invariance of the Leray-Schauder degree and the symmetry of f, the local degrees satisfy the equation $(s \in (0, \varepsilon))$

$$\begin{aligned} d(-\lambda_s) &= d(\lambda_s) + \deg\big(f_{p(\lambda_s)}, u(\lambda_s) + v_s\big) + \deg(f_{p(\lambda_{-s})}, u(\lambda_{-s}) + v_{-s}) \\ &= d(\lambda_s) + 2 \cdot \deg(f_{p_1}, u_1). \end{aligned}$$

This implies $|\deg(g_{p_1}, u_1)| = |\deg(f_{p_1}, u_1)| = 1$ and we can apply the global implicit function theorem to get a four-dimensional subset Z_0 of $f^{-1}(0) \cap P \times X_\nu$. Set $Z_1 := \overline{Z}_0 - M \subset Z - M$. We know that the homomorphism $i_0^* : H^4(E^+ - \Sigma) \to H^4(Z_0^+)$ is not zero. Now, E^+ can be considered as the quotient space

$$E^+ = (P \times X_\nu \cup \{\infty\}) / (\partial P \times X_\nu \cup \{\infty\})$$

and similarly

$$Z_0^+ = (Z_1 \cup M \cup \{\infty\}) / ((Z_1 \cup M \cup \{\infty\}) \cap (\partial P \times X_\nu \cup \{\infty\})).$$

By the continuity and the excision property of H^* (cf. [13], Theorem 6.6.5) we have

$$\begin{aligned} H^*(E^+ - \Sigma) &\cong H^*(P \times X_\nu \cup \{\infty\} - \Sigma, \partial P \times X_\nu \cup \{\infty\}) \\ &\cong H^*(P \times X_\nu - \Sigma, \partial P \times X_\nu) \end{aligned}$$

and

$$H^*(Z_0^+) \cong H^*(Z_1 \cup M \cup \{\infty\}, (Z_1 \cup M \cup \{\infty\}) \cap (\partial P \times X_\nu \cup \{\infty\}))$$
$$\cong H^*(Z_1 \cup M, (Z_1 \cup M) \cap (\partial P \times X_\nu)).$$

Thus the inclusion

$$i_1 : (Z_1 \cup M, (Z_1 \cup M) \cap (\partial P \times X_\nu)) \hookrightarrow (P \times X_\nu - \Sigma, \partial P \times X_\nu)$$

induces a non-zero map

$$i_1^* : H^4(P \times X_\nu - \Sigma, \partial P \times X_\nu) \to H^4(Z_1 \cup M, (Z_1 \cup M) \cap (\partial P \times X_\nu)).$$

$i_1^* \neq 0$ implies that i_1 is not nullhomotopic. □

It remains to prove Theorem (5.3). This is essentially a one-parameter result. Suppose there exists $p_- \in K_-$ with $p_- \notin \pi(Z_0)$. Let $p_+ \in K_+$ be arbitrary. Choose a path $\omega = (p, u) : [-1, 1] \to M$ with $p(-1) = p_-$, $p(+1) = p_+$ and such that the image of p intersects $\pi(\overline{Z}_0 \cap M)$ only in $p(0) = p_0 \in P - P_2$. We do not assume that the image of p is contained in $P - P_2$ since neither K_+ nor K_- need be contained in $P - P_2$. Also, we allow ω to intersect the singular set S outside of $\overline{Z}_0 \cap M$. But we may assume that (H1) and (H2) are satisfied near $\omega(0)$. Then, we consider the map

$$g : [-1, 1] \times X_\nu \to Y_\nu, \qquad g(\lambda, v) := f(p(\lambda), u(\lambda) + v).$$

Obviously, $g(\lambda, 0) = 0$ and $\lambda_0 = 0$ is a global bifurcation point (bifurcation from a simple eigenvalue). Let Z_0' be the connected component of $(0,0)$ in $\big(g^{-1}(0) - [-1,1] \times \{0\}\big) \cup \{(0,0)\}$. The map $(\lambda, v) \mapsto \big(p(\lambda), u(\lambda) + v\big)$ maps Z_0' to Z_0 because the image of p intersects $\pi(\overline{Z}_0 \cap M)$ only in $p(0) = p_0$. Therefore, Z_0' is bounded and $\overline{Z_0'} \cap [-1, 1] \times \{0\} = \{(0, 0)\}$. Furthermore, $Z_0' \cap \{-1\} \times X_\nu = \emptyset$ since $p_- \notin \pi(Z_0)$. Then the global bifurcation result of Rabinowitz [12] or Magnus [11] implies $Z_0' \cap \{+1\} \times X_\nu \neq \emptyset$. This just says $p_+ \in \pi(Z_0)$. □

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