Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 2, 1993, 91–104

CONNECTEDNESS OF THE BRANCH OF POSITIVE SOLUTIONS OF SOME WEAKLY NONLINEAR ELLIPTIC EQUATIONS

E. N. DANCER

(Submitted by K. Gęba)

Dedicated to the memory of Juliusz Schauder

In this paper, we consider branches of positive solutions of

$$\begin{aligned} -\Delta u &= \lambda f(u) && \text{in } D, \\ u &= 0 && \text{on } D, \end{aligned}$$

for $\lambda > 0$ where D is a domain in R^2. We assume that D has a double symmetry and convex (though the last assumption can be weakened). We prove for rather general f that the branch of non-trivial positive solutions of (1) forms smooth curves. This generalizes slightly a result of Kielhofer [16]. Moreover, for a wide variety of possible non-negative reasonably smooth f's, we prove that the branch of non-trivial positive solutions of (1) forms a *connected* curve. This seems to be the first result of this type for a true partial differential equation. Previously, results of this type for rather general f have only been proved for D a ball, where the problem reduces to an ordinary differential equation. We stress that our assumptions on f allow the branch of positive solutions to have many changes of direction. Note that our assumptions on D are not superfluous to our results. Indeed, we constructed examples in [6] for rather nice f's on doubly symmetric star shaped domains where the branch of non-trivial positive solutions have true secondary bifurcations and examples on a star shaped domain with a single line of symmetry where the positive solutions are not connected even though, for the same f, the corresponding branch for D a ball is connected. It is easy to deduce from [4] that this last behaviour continues to hold for some 'generic' smooth domains.

The basic ideas of our proof are to deform D to a ball by using domain variation techniques and to avoid non-compactness difficulties by proving local uniform (in D) estimates for small or large solutions or for solutions with λ small or large. In many cases, for this last step, we use ideas from [5]. We can justify this procedure in many cases. However, there are some cases where we have not succeeded in doing this. One important case is where f grows exponentially as in the Gelfand equation, another is the case where f changes sign for positive y (as in the work of Hess [15] or Dancer [9]). However, our results do cover a wide variety of cases. It seems that this technique is also useful in other problems. For example, it can be used for higher dimensional domains close to balls, or for perturbations of our symmetric domains in two dimensions.

I should like to thank Professor P. Srikanth for some useful conversations.

1. The Main Results

In this section, we prove our main results. We first improve very slightly a result of Kielhofer. We consider a bounded C^3 domain D in R^2 containing $(0,0)$ such that D is invariant under the symmetries $(x,y) \to (x,-y)$ and $(x,y) \to (-x,y)$ and such that $\partial D \cap \{(x,y) : x \geq 0,\ y \geq 0\} = \{(x,h(x)) : 0 \leq x \leq a\}$ where h is decreasing. We say that a domain of this type is of type R.

PROPOSITION. *Assume that f is C^1, $f(0) \geq 0$ and D is of type R. Then $W = \{(u,\lambda) \in C(\overline{D}) \times (0,\infty) : u$ is a non-trivial positive solution of $(1)\}$ is a smooth 1-manifold without boundary (not necessarily connected).*

PROOF. It suffices to prove that for every $(u.\lambda) \in W$, the map Z which sends (h,τ) to $-\Delta h - \lambda f'(u)h - \tau f(u)$ maps $\{u \in W^{2,p}(D) : u = 0 \text{ on } \partial D\} \times R$ onto $L^p(D)$ (where $p > 1$), because we can then apply a theorem of Amann [2]. Note that since $f(0) \geq 0$, the maximum principle ensures that every positive solution w has non-zero normal derivative on ∂D and hence any solution C^1 close to w will also be positive in D. If the equation

$$\begin{aligned} -\Delta h &= \lambda f'(u)h \qquad && \text{in } D, \\ h &= 0 && \text{on } \partial D, \end{aligned}$$

has only the trivial solution, the ontoness is clear. Thus, it suffices to assume that (2) has a non-trivial solution. In this case, we will prove that (2) has a one-dimensional kernel spanned by $\widehat{h}$. By standard theory, it follows that the map

$h \to -\Delta h - \lambda f'(u)h$ has range $V = \{w \in L^p(D) : \int_D w\widehat{h} = 0\}$. Note that this subspace has codimension 1 in $L^p(D)$. Hence our map Z will be onto if $\int_D f(u)\widehat{h} \neq 0$ (since $f(u)$ will not be in V). Hence it suffices to prove that the kernel Y of (2) is at most one-dimensional and $\int_D f(u)\widehat{h} \neq 0$ if $\widehat{h} \in Y$. We in fact prove that $\int_D f(u)p \neq 0$ whenever p is a non-trivial member of Y. This suffices, because any linear functional (that is, with codomain R) on a space of dimension more than one must have a non-trivial kernel. Hence our last result implies that Y is one-dimensional. Hence it suffices to prove that $\int_D f(u)p \neq 0$ if $p \in Y \setminus \{0\}$.

To prove this, note that the argument in Step 1 of the proof of Theorem 5 in [4] shows that every $p \in Y$ is even in x and y and, as in part of the proof of Step 2 there, if $p \in Y \setminus \{0\}$, the closure of $\{x \in D : p(x) = 0\}$ does not intersect ∂D. (In [4], a specific nonlinearity is used but this part of the argument does not use the form of the nonlinearity.) There is also a slight gap in [4] which is fixed in the following lemma. Hence $\frac{\partial p}{\partial n}$ has fixed sign on ∂D and, by the maximum principle, is non-zero on ∂D. Now, as in [16], $r\frac{\partial u}{\partial r}$ is a solution of

$$-\Delta v - \lambda f'(u)v = 2\lambda f(u)$$

in D (but not satisfying the boundary condition). We multiply this equation by p, integrate over D and use that p is a solution of (2). We eventually find that

$$\int_D 2\lambda f(u)p = \int_{\partial D} r\frac{\partial u}{\partial r}\frac{\partial p}{\partial n}.$$

Now, $r\frac{\partial u}{\partial r} < 0$ on ∂D by the Gidas-Ni-Nirenberg theorem [11]. By this and our earlier comments on $\frac{\partial p}{\partial n}$ on ∂D, the right hand side is non-zero and hence our claim follows.

This result has a variant if D is convex but does not have the symmetries. It can be shown that the branch is locally a smooth curve near (u_0, λ_0) if any non-trivial solution h of (2) (for $u = u_0$, $\lambda = \lambda_0$) has the property that the closure (in $\overline{D}$) of the nodal lines in D intersect ∂D in at most two points (for example if h is a second eigenfunction). It is partly based on some unpublished work of C. S. Lin (in particular a change of origin).

There is a slight gap in the argument above (and also in [4], [8], [10] and [16].) The estimates for the zero set in [13] and [14] are only proved in the interior of D and not near the boundary. To overcome this, we need the following lemma.

LEMMA. *Assume that the above assumptions hold and h is a non-trivial solution of (2) vanishing on ∂D. Then, Green's theorem is valid for smooth functions for any component of $\{x \in D : h(x) = 0\}$.*

PROOF. Assume $x_0 \in \partial D$. Without loss of generality, we may assume that $x_0 = 0$ and that $n(x_0)$, the unit normal to ∂D at x_0, is e_1. Near 0, we can write

∂D as $\{te_2 + s(t)e_1\}$ where s is C^3 (since ∂D is C^3). We choose a better set of coordinates. Let $w(t) = te_2 + s(t)e_1$. Let $n(t) = n(w(t))$. Then, near zero, we can use $(t, s) \to w(t) + sn(t)$ as a local system of coordinates (as follows easily from the implicit function theorem). This is a C^2 change of coordinates. On $s = 0$, the derivative of the change of coordinates is diagonal since $w'(t)$ is perpendicular to $n(t)$. We now see what form our equation takes in the new coordinates. It will be of the form

$$a_{ij}(t,s)\frac{\partial^2 h}{\partial t_i \partial t_j} + b_i(t,s)\frac{\partial h}{\partial t_i} + c(t,s)h = 0$$

where $t_1 = t,\ t_2 = s,\ A = (a_{ij})$ is diagonal on $s = 0,\ a_{ij}$ are C^1, b_i are continuous and c is continuous. To see this, we use the formula for a change of coordinates in Saut and Teman ([19], equation (4.10)) which is easily proved directly (most easily by using the weak form of the equation). Note that the top order coefficients are of the form $(\det T')[(T')^t T']^{-1}$ where T' is the derivatives of the change of coordinates. Hence the top order coefficients are diagonal on $s = 0$ (since T' is diagonal there). Note that in the new coordinates the boundary is $s = 0$ and h is C^2 on $s \geq 0$ by standard regularity theory applied to (2). We extend h across $s = 0$ by extending it to be odd in s. This is reasonable since $h = 0$ when $s = 0$. We extend c to be even in s, b_1 to be even in s, b_2 to be odd in s, a_{11} and a_{22} to be even in s and a_{12} and a_{21} to be odd in s. Then it is easy to see that the extended h satisfies (3) in a neighbourhood of $(0,0)$. The coefficients are as regular as before except that b_2 need not be continuous. In particular, the a_{ij} are C^1. We now can obtain the desired result by applying the results of Hardt and Simon [13] (including the remarks on p. 505) to estimate Hausdorff measures of zero sets and by then using the arguments in the proof of Step 1 of Lemma 2 in [8].

REMARKS. 1. The result can be improved to show that the zero sets consist of curves C^1 up to the boundary if D has C^∞ boundary. (We use the ideas in [18].) Presumably one could prove this under much weaker assumptions on ∂D.

2. The above argument is valid in all dimensions with only minor modifications. In general, the change of variable T' is not diagonal but has one column orthogonal to the other columns (when $s = 0$). This ensures that $a_{nj}(\tilde{t}, 0) = 0$ for $1 \leq j \leq n-1$ where $\tilde{t} = (t_1, \ldots, t_{n-1})$ locally coordinatizes ∂D. Otherwise the proof is as before.

3. The above argument can also be applied for more general second order elliptic operators, in any dimension, whose top order coefficients are C^1. We choose coordinates so that $a_{ij}(0) = \delta_{ij}$ and then use the implicit function theorem to choose $n(t,s)$ so that the coordinates $t + sn(t,s)$ have similar properties to above (where $t \in \partial D$).

4. Note that the reflection trick is trivial across flat boundaries.

We now obtain our main result. Assume that $a > 0$ (possibly $+\infty$) and that f is C^1 on $[0, a]$ and $f(y) > 0$ on $(0, a)$. We also assume that D is of type R and that the following two conditions hold:

(i) $f(0) > 0$ or $f'(0) = 0$ or there is a $p > 1$ such that $y^{1-p} f'(y) \to \widetilde{q} \in (0, \infty)$ as $y \downarrow 0$;

(ii) one of the following holds:

(α) $a < \infty$, $f(a) = 0$, there is an $\varepsilon > 0$ such that $f'(y) \leq 0$ on $(a, a - \varepsilon)$,

(β) $a = \infty$ and $f'(y) \to c \in (0, \infty)$ as $y \to \infty$,

(γ) $a = \infty$, $f(y) \to C \in (0, \infty)$ as $y \to \infty$ and $yf'(y) \to 0$ as $y \to \infty$,

(τ) $a = \infty$ and there exists a $q > 1$ such that $f'(y)/y^{q-1} \to c \in (0, \infty)$ as $y \to \infty$,

(μ) $a = \infty$ and there exists a $q < 1$ such that $f'(y)/y^{q-1} \to c$ as $y \to \infty$.

Note that these conditions are satisfied by a wide variety of f's.

Let $W = \{(u, \lambda) : \lambda > 0, (u, \lambda)$ is a non-trivial positive solution of (1) with $\|u\|_\infty < a\}$.

THEOREM 1. *Under the above assumptions, W is a connected manifold.*

PROOF. The result is proved by continuing D to the unit ball through a smooth deformation D_t (where $D_1 = D$ and D_0 is the unit ball with centre 0 and where D_t satisfies our earlier assumptions for each t). Note that it is easy to construct such a deformation. We know that W_t is a smooth manifold for each t (by the proposition). Here W_t is the analogue of W when D is replaced by D_t. We use a continuation argument to prove W_t is connected. The proof involves a good deal of checking that certain standard arguments can be done uniformly in t. These are easy, but very tedious, and hence we only emphasize the main points. Most of the difficulties are caused by the non-compactness of W_t. We have to keep careful control of the non-compact 'ends' of the 1 manifold W_t.

STEP 1. When $t = 0$, the Gidas, Ni and Nirenberg theorem ensures that every solution of (1) in W_0 is radially symmetric. Thus they are solutions of an ordinary differential equation. We prove that W_0 is connected by showing that it can be parametrized by $(0, a)$ in a continuous fashion. Let $w(r, s)$ denote the unique radial solution of $-\Delta u = f(u)$ satisfying $w(0, s) = s$ and $w_1'(0, s) = 0$. Then w is jointly continuous. Since $f(x) \geq 0$ on $[0, a]$, it is easy to see from the ordinary differential equation that, if $0 \leq s \leq a$, then w is decreasing in r as long as w is non-negative and that $w_1'(\widetilde{r}, s) < 0$ if $\widetilde{r}$ is the first positive zero of w. Note that w must have such a zero since otherwise $0 \leq w(r, s) \leq a$ on $[0, \infty)$ and $-\Delta w = f(w) \geq 0$ on R^2 and hence w is a bounded subharmonic function on R^2. It is well known that this implies that w is constant (which must be s). This is impossible since $f(s) > 0$. Now $\widetilde{r}$ is a continuous function of s. It is easy to see by a rescaling of r that

$$W_0 = \{(w(\widetilde{r}r, s), (\widetilde{r})^2) : 0 < s < a\}$$

and hence W_0 is connected since $\widetilde{r}$ is a continuous function of s and w is a continuous function of r and s.

STEP 2. *Next we prove that the branches are continuous in t locally.*

We basically work with the space $L^\infty(R^2) \times R$, where we extend a solution on D_t to R^2 by extending it to be zero outside D_t. More precisely, assume that $t_0 \in [0,1]$ and $(u_0, \lambda_0) \in W_{t_0}$. We only consider the more difficult case where $-\Delta - \lambda_0 f'(u_0)I$ with Dirichlet boundary conditions on D_{t_0} is not invertible. We prove that there exist solutions (u^t, λ^t) on D_t near (u_0, λ_0), depending continuously on t, a complement M_0 to span $\{h_0\}$, $\delta > 0$ and functions $\psi_t(\cdot) : (-\delta, \delta) \to M_0$ and $\phi_t(\cdot) : (-\delta, \delta) \to R$ depending continuously on t and s (jointly) such that $\psi_t(0) = 0$, $\phi_t(0) = 0$ and $(u^t + sh_t + \psi_t(s), \lambda^t + \phi_t(s))$ is a solution of (1) (for domain D_t) and these are the only solutions of (1) (on D_t) in a ball with centre (u_0, λ_0) and of radius ε in $L^\infty(R^2) \times R$ (where ε is independent of t). Here h_0 is the unique normalized eigenfunction of $-\Delta - \lambda_0 f'(u_0)I$ (on D_{t_0}) corresponding to a zero eigenvalue and M_0 is the complement determined by the spectral projection. There are two ways to prove this result. Firstly, we can use the method of Saut and Teman [19] to reduce to a problem on a fixed domain depending quite smoothly on t (since D_t depends rather smoothly on t). We then mimic the usual proof of the Amann theorem [2] (which reduces to the contraction mapping theorem). We have extra terms in the equation due to the t variation but it is easy to see that these are small and satisfy a small Lipschitz condition (as maps from $W^{2,2}(D_{t_0})$ to $L^2(D_{t_0})$). The details are very easy but a little tedious. As in [4], it is convenient to work with the equation where f is truncated for $|y|$ large. The alternative proof which holds for much more general domain dependence is a variant of the proof of Theorem 3 in [4]. (In fact it could be used to generalize the result there.) We sketch it briefly. We approximate u_0 in the $W^{1,2}$ norm by $\widetilde{u}^n$ where $\widetilde{u}^n \in C_0^\infty(D_{\lambda_0})$ and the $\widetilde{u}^n$ are uniformly bounded. The argument in Step 3 of the proof in Theorem 1 in [4] then shows that, if this approximation is close enough, then for t near t_0, $-\Delta - \lambda_0 f'(\widetilde{u}_n)$ on D_t has a unique small eigenvalue $\widetilde{\lambda}_n(t)$ with corresponding normalized eigenfunction $\widetilde{h}_n(t)$ and no other eigenvalue is small. Let $W_n(t)$ be the orthogonal complement of $\widetilde{h}_n(t)$ (in $L^2(\Omega_t)$) and let P_n be the orthogonal projection onto $W_n(t)$. By the above comments on the eigenvalues, $(-\Delta - \lambda_0 f'(\widetilde{u}_n)I)^{-1}$ (considered on D_t) is uniformly bounded as a map of $W_n(t)$ into itself (for the L^2 norm). As in Step 2 of the proof of Theorem 3 in [4], one deduces that $(-\Delta - \lambda_0 f(\widetilde{u}_n)I)^{-1}$ is uniformly bounded as a map of $W_n(t) \cap L^\infty(D_t)$ into itself (for the L^∞ norm). The remainder of the proof is a contraction mapping argument rather similar to the proof of Step 3 of Theorem 3 in [4]. We look for solutions of the form $(u_0 + \alpha h_n(t) + \psi_n(t,\alpha),\ \phi_n(t,\alpha))$. There are a couple of extra terms in the analogue of equation (12) in [4] but these are easily seen to be small (with small Lipschitz constant) if n is large. Note that (u^t, λ^t) is the solution for $\alpha = 0$. The parametrization above is slightly different to the one given in the statement of this step. However, since ψ_n satisfies a small Lipchitz condition in α, it is easily seen that they are equivalent.

STEP 3. *Next we prove that, if C is a compact connected subset of a component of W_t, then there are neighbourhoods U, V of C in $C(B) \times R$ with $U \subseteq V$ such that any two points of $U \cap W_s$ lie in the same component of $V \cap W_s$ for all s near t.*

Here B is a ball of large radius containing every D_t. This shows that there are no difficulties semi-locally. To see this, we first note that, if $k > 0$ and if s is close to t, then any point of $\overline{W}_s \cap \{(u,\lambda) \in C(B) \times R : \|u\|_\infty + |\lambda| \le k\}$ is close to W_t in $C(B) \times R$. This follows easily by arguments similar to those in the proof of Theorem 1(ii) in [4] and Remark 3 after it. By this result, the results of the previous paragraph, and by a simple compactness argument, we see that there exist positive numbers δ_i, $i = 1, \ldots, k$, neighbourhoods S_i^1, S_i^2, $i = 1, \ldots, k$, in $E \times R$, jointly continuous functions $Z_s^i : (-\delta_i, \delta_i) \to E \times R$ such that the range of Z_s^i is contained in W_s, the range of Z_t^i intersects C, Z_s^i is $1-1$ for fixed i and s and

$$W_s \cap U_1 \subseteq \bigcup_{i=1}^{k} \mathrm{Range}(Z_s^i), \quad W_s \cap S_i^2 \supseteq \mathrm{Range}(Z_s^i) \supseteq W_s \cap S_i^1 \quad \text{for } 1 \le i \le k$$

for all s near t. Here $E = C(B)$ as before and U_1 is a neighbourhood of C in $C(B) \times R$. Now C is a compact proper connected subset of the connected 1 manifold W_t and hence is homeomorphic to an interval $[a, b]$. Hence we can regard the Z_s^i as parameterized by open subsets P_i of R intersecting $[a, b]$ (by using $Z_s^i \circ (Z_t^i)^{-1}$) where the open subsets are independent of i and clearly cover $[a, b]$, (since $\mathrm{Range}(Z_t^i)$, $i = 1, \ldots, k$, cover C). Hence we see that $\bigcup_{i=1}^k \mathrm{Range}(Z_s^i)$ is connected because we can clearly move between any two points by simply changing from Z_s^i to Z_s^j whenever the P_i intersect. (Recall that the range of Z_s^i is connected for fixed s and $\mathrm{Range}\, Z_s^i$ exhausts $W_s \cap S_i^1$.) We now define $U = \bigcup_{i=1}^k S_i^1$ and $V = \bigcup_{i=1}^k S_i^2$ and the result follows easily.

STEP 4. *Let*

$$\widetilde{W}_{\varepsilon,t} = \{(x,\lambda) \in W_t : \|x\|_\infty < \varepsilon \text{ or } \|x\|_\infty > a_\varepsilon \text{ or } \lambda > \varepsilon^{-1} \text{ or} \lambda < \varepsilon\}.$$

Here $a_\varepsilon = \varepsilon^{-1}$ if $a = \infty$ and equals $a - \varepsilon$ if $a < \infty$. If $t \in [0,1]$, we will prove, in Step 5, that there exist ε, $\delta > 0$ such that for each s with $|s - t| \le \delta$ each component of $\widetilde{W}_{\varepsilon,s}$ intersects $\|x\|_\infty = \varepsilon$ or $\|x\|_\infty = a_\varepsilon$ or $\lambda = \varepsilon$ or $\lambda = \varepsilon^{-1}$.

Moreover, exactly two components of $\widetilde{W}_{\varepsilon,s}$ are not compact in $L^\infty(R^2) \times (0,\infty)$ (and these are never close to each other). To see that this suffices to complete the proof, we let $\widetilde{A} = \{s \in [0,1] : W_s$ is connected$\}$.

Since $\widetilde{A}$ is non-empty, it suffices to prove that $\widetilde{A}$ is open and closed in $[0,1]$.

Assume W_t is connected. The result above implies that, if ε, $\delta > 0$ are as above, then any component of W_s for $|s-t| \le \delta$ intersects $\widehat{B}_{\varepsilon,s} \equiv \{(x,\lambda) \in C(D_s) \times R : \varepsilon \le \|x\|_\infty \le a_\varepsilon,\ \varepsilon \le \lambda \le \varepsilon^{-1}\}$. Thus, to prove that W_s is connected for such s, we only need to prove that, if $|s-t| \le \delta$ and (u_1,λ_1), $(u_2,\lambda_2) \in W_s \cap \widehat{B}_{\varepsilon,s}$, then (u_1,λ_1), (u_2,λ_2) lie in the same component of W_s. By a simple compactness argument (cp. the proof of Theorem 1(ii) in [4]), $W_s \cap \widehat{B}_{\varepsilon,s}$ is close to $W_t \cap \widehat{B}_{\varepsilon,t}$ in the sup norm if s is close to t. (Technically we first prove closeness in an L^p or Orlicz norm, then use interior regularity results to prove closeness in the sup norm on compact sets in the interior. Finally barriers can easily be used to show that solutions are uniformly small close to the boundary and hence we have the uniform convergence. Note that barriers are discussed on p. 98–101 of [12].) In particular, it follows that (u_1,λ_1) and (u_2,λ_2) are close to points of $W_t \cap \widehat{B}_{\varepsilon,t}$ if s is close to t. Suppose our claim is false. Then there exist $t_n \to t$ and (u_1^n,λ_1^n), (u_2^n,λ_2^n) in $W_{t_n} \cap \widehat{B}_{\varepsilon,t_n}$ being in different components of W_{t_n}. By our earlier remarks and by taking subsequences, we see that we can assume that $(u_1^n,\lambda_1^n) \to (\widehat{u}_1,\widehat{\lambda}_1)$ and $(u_2^n,\lambda_2^n) \to (\widehat{u}_2,\widehat{\lambda}_2)$ in $L^\infty(R^2) \times R$ as $n \to \infty$. Now there is a compact connected subset $\widehat{C}$ of W_t containing $(\widehat{u}_1,\widehat{\lambda}_1)$ and $(\widehat{u}_2,\widehat{\lambda}_2)$. (This follows easily that W_t is a connected 1-manifold and hence is homeomorphic to R.) Since $\widehat{C}$ is compact, Step 3 now implies that (u_1^n,λ_1^n) and (u_2^n,λ_2^n) lie in the same component of W_{t_n} for large n. This gives the required contradiction. Hence W_s is connected for s near t and $\widetilde{A}$ is open.

To prove that $\widetilde{A}$ is closed assume by way of contradiction that W_{s_n} is connected, W_t is not and $s_n \to t$ as $n \to \infty$. Since W_t is not connected, we can use standard results on separating compact sets (as in Whyburn [20]) to show that $W_t = K_1 \cup K_2$ where K_1 and K_2 are closed and disjoint and the two non-compact (in $L^\infty(R^2) \times (0,\infty)$) components T_1^t, T_2^t of $\widetilde{W}_{\varepsilon,t}$ do not both lie in K_1 (or both lie in K_2). (Technically, we apply separation results to a two point compactification of W_t.) We easily see that K_1 and K_2 are a positive distance apart. We may assume $T_1^t \subseteq K_1$. By our earlier remarks any point of W_{s_n} for n large will be close to K_1 or to K_2 or will lie in one of the two non-compact components of W_{ε,s_n}. This gives a separation of W_s. (One part of the separation will be the points of W_{s_n} near K_1 or near $T_1^{s_n}$ and the other is defined analogously.) This separation contradicts the connectedness of W_{s_n} and hence our claim follows. Thus $\widetilde{A}$ is closed. This completes Step 4.

STEP 5. *Thus the proof of the theorem reduces to proving the results on the structure of $\widetilde{W}_{\varepsilon,s}$ locally uniformly in s mentioned at the beginning of Step 4.*

This basically reduces to checking that a number of standard arguments can be done locally uniformly in s. We first dispose of the easy cases. It is easy to see that $(-\Delta)^{-1}$ is uniformly bounded as a map of $L^\infty(D_s)$ into itself uniformly in s. (We will meet more complicated cases a little later.) It follows easily by a contraction mapping argument that, if $f(0) > 0$, then there is an $\widehat{\varepsilon} > 0$ independent of s such that $\{(x, \lambda) \in W_s : \|x\|_\infty \leq \widehat{\varepsilon}\}$ is contained in $\{(x_s(\lambda), \lambda) : 0 \leq \lambda \leq \widehat{\varepsilon}\}$ where x is jointly continuous in s and λ. In addition, if $f(y) \leq Ay + \widetilde{c}$ for $y \leq B$, then, uniformly in s, $W_s \cap \{(x, \lambda) : \widehat{\varepsilon} \leq \|x\|_\infty \leq B,\ \lambda \leq \widetilde{\varepsilon}\}$ is empty for suitable $\widetilde{\varepsilon}$. (The last statement is valid even if $B = \infty$ or if $f(0) = 0$.) This disposes of all the solutions with λ small except for large solutions when f is superlinear. We return to this case later. Moreover, if $f(0) > 0$, it is easy to see that the only solutions (x, λ) with $\|x\|_\infty$ small have λ small (once again uniformly in s).

We now consider the solutions with $\|x\|$ small when $f(0) = 0$ and $f'(0) > 0$. In this case, the first and second eigenvalues λ_1^s, λ_2^s of $-\Delta$ on D_s (for Dirichlet boundary conditions) as well as the first (normalized) eigenfunction h_1^s depend continuously on s. It follows easily that, for λ close to $(f'(0))^{-1}\lambda_1^s$, $(-\Delta - \lambda f'(0)I)^{-1}$ is uniformly bounded (in s) as a map of $(h_1^s)^\perp$ into itself for the L^2 norm on D_s. We can then argue as in Step 2 of the proof of Theorem 3 in [4] to deduce that this same operator is uniformly bounded in s as a map of $(h_1^s)^\perp \cap L^\infty(D_s)$ into itself for the L^∞ norm. It is now easy to check the usual proof of the Crandall-Rabinowitz bifurcation theorem [2] holds uniformly in s in the space $L^\infty(D_s)$. More precisely, one finds that there exist $\widetilde{\varepsilon} > 0$ and jointly continuous functions $\phi_s : (-\widetilde{\varepsilon}, \widetilde{\varepsilon}) \to R$ and $\psi_s : (-\widetilde{\varepsilon}, \widetilde{\varepsilon}) \to (h_1^s)^\perp \cap L^\infty(D_s)$ with $\phi_s(0) = f'(0)^{-1}\lambda_1^s$, $\psi_s(0) = 0$ such that $(\alpha(h_1^s + \psi_s(\alpha)), \phi_s(\alpha))$ are positive solutions of (1) on D_s for $0 < \alpha < \widetilde{\varepsilon}$ and these are the only small positive solutions close to $(0, (f'(0))^{-1}\lambda_1^s)$ where what is meant by close is uniform in s. Moreover, these are the only solutions (x, λ) which have $\|x\|_\infty$ small. This follows easily because, if $A_1 y \leq y^{-1} f(y) \leq A_2 y$ for $0 \leq y \leq B$ (where $A_1 > 0$), a simple comparison argument shows any positive solution (x, λ) with $\|x\|_\infty \leq B$ satisfies $\lambda A_1 \leq \lambda_s^1 \leq \lambda A_2$. This argument also shows that, if $f'(0) > 0$ and if $f(y) > 0$ for $y > 0$, then the only positive solutions with λ large are large in the uniform norm. (Once again, if $a < \infty$, this becomes that $\|u\|_\infty \geq a - \varepsilon$ if $(u, \lambda) \in W_s$.)

Moreover, if $f'(y) \to c \in (0, \infty)$ as $y \to \infty$, one can use similar arguments to show that large solutions occur for λ close to $(f'(\infty))^{-1}\lambda_1^s$ and form a one parameter family bifurcating from infinity parametrized by $P_s x$ where P_s is the orthogonal projection onto the span of h_1^s (and this holds uniformly and continuously in s). The proof of this is very similar to the case in the previous paragraph except that we replace the Crandall-Rabinowitz argument by the argument of the author [7]. One difference is that $\alpha \in (\tilde{\varepsilon}^{-1}, \infty)$ instead of $(0, \varepsilon)$. The uniformity in s is easy to check as in the previous paragraph. In this case, our earlier arguments imply that there cannot be large solutions for λ small. If $-\Delta u_n = \lambda_n f(u_n)$ on D_{s_n}, $\|u_n\|_\infty \to \infty$, $\lambda_n \to \overline{\lambda}$ as $n \to \infty$ and $s_n \to \overline{s}$, let $v_n = (\|u_n\|_\infty)^{-1} u_n$. Then one easily sees that $-\Delta v_n - \overline{\lambda} v_n \to 0$ as $n \to \infty$ in $C(D_{s_n})$ and a simple limit argument (similar to those in [4]) easily implies that $\overline{\lambda} = \lambda_1^{\overline{s}}$. Thus large solutions could only occur for λ near λ_1^s or λ large. If $f'(0) > 0$ or $f(0) > 0$, $f(y) \geq \mu y$ on R^+ where $\mu > 0$ an earlier argument implies that there are no positive solutions for λ large. If $f'(0) = 0$, it can be proved by a blowing up argument similar to those below that there can be no non-small positive solutions with λ large. We do not give the details because more complicated cases occur later.

The remaining cases are handled by showing that arguments in [5] can be done locally uniformly in s. It is assumed that the reader has a copy of [5] available. Note that blowing up arguments are often easier than those in [5] because the maximum of a solution always occurs at 0 (by Gidas-Ni-Nirenberg [11]) and thus the maximum never occurs close to the boundary. This makes it very easy to generalize many of the results in [5] to our case where the domain is varying. In particular, we never have to do boundary blow ups.

We consider in some detail the case of large solutions when $f(y) \sim y^p$ as $y \to \infty$ (that is, $f(y) = y^p g(y)$ for large y where $g(y) \to c \in (0, \infty)$ as $y \to \infty$). Assume that u_n are positive solutions of $-\Delta u_n = \lambda_n f(u_n)$ on D_{s_n} such that $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Since $f(y) \leq Ay^p + B$ on R^+ and since $(-\Delta)^{-1}$ is bounded as a map of $L^\infty(D_{s_n})$ into itself (uniformly in s), it follows by simple estimations that there is an $\varepsilon_0 > 0$ independent of n such that $\varepsilon_0 \leq \lambda_n (\|u_n\|_\infty)^{p-1}$ for all n. Let $v_n = (\|u_n\|_\infty)^{-1} u_n$. Then

$$-\Delta v_n = \lambda_n (\|u_n\|_\infty)^{p-1} \frac{f(\|u_n\|_\infty v_n)}{(\|u_n\|_\infty)^p}.$$

Now it is easy to check that $f(\|u_n\|_\infty v_n)/(\|u_n\|_\infty)^p - c v_n^p \to 0$ uniformly as $n \to \infty$. Hence if a subsequence of $\lambda_n (\|u_n\|_\infty)^{p-1}$ tends to infinity as n tends to infinity, we can repeat the blowing up argument in [5], p. 441, to obtain a bounded positive solution of $-\Delta v = v^p$ on R^2. (We are using the remark above on why the variation of the domain does not affect this argument. Note that when we blow up close to the maximum we are always looking at points close to zero and these points lie in all the domains.) Such a solution would have to be a bounded superharmonic function on R^2 and thus constant. This is easily seen to be impossible and hence $\lambda_n (\|u_n\|_\infty)^{p-1} \leq K_1$ for all n. Thus $\lambda_n \to 0$ as $n \to \infty$ and we can write $u_n =$

$\lambda_n^{-1/(p-1)} w_n$ where $K_3 \le \|w_n\|_\infty \le K_4$ for all n. Now w_n satisfies

$$-\Delta w_n - \lambda_n^{p/(p-1)} f(\lambda_n^{-1/(p-1)} w_n) = 0 \quad \text{on} \quad D_{s_n}$$

and hence, since w_n is uniformly bounded, we see much as before that $-\Delta w_n - cw_n^p \to 0$ in $L^\infty(D_{s_n})$ as $n \to \infty$. Much as earlier (or as in [4]) we now deduce that w_n (or at least a subsequence) converges uniformly to a non-trivial non-negative solution w of $-\Delta w = cw^p$ in $D_{\overline{s}}$, $w = 0$ on $\partial D_{\overline{s}}$, where $\overline{s}$ is the limit of the s_n. (As before we first prove convergence in L^r for large r, then uniform convergence on compact subsets of the interior and finally use barriers to obtain uniform convergence near the boundary.) Since (5) has a unique non-negative non-trivial solution $\overline{w}_{\overline{s}}$ (by [4]), it follows that $w_n \to \overline{w}_{\overline{s}}$ as $n \to \infty$. Conversely, we prove that there are $\varepsilon_1, \varepsilon_2 > 0$ such that if $\lambda \le \varepsilon_1$, (1) has a unique positive solution of the form $\lambda^{-1/(p-1)}(\overline{w}_s + h(s,\lambda))$ where $\|h(s,\lambda)\|_\infty \le \varepsilon_2$. Moreover, this solution depends continuously on s and λ. This now follows by standard theory for degree theory and contraction mappings if we note the following. Firstly, the mapping $h \to -\Delta h - cp(\overline{w}_s)^{p-1}h$ (with Dirichlet boundary conditions on D_s) is invertible by the proof of Theorem 5 in [4]. Secondly, as earlier, we can prove a uniform estimate for the inverse in $L^\infty(D_s)$. Thirdly, $\lambda^{p/(p-1)} f(\lambda^{-1/(p-1)} y) = cy^p + g(y,\lambda)$ where g and $\frac{\partial g}{\partial y}(y,\lambda)$ are small on bounded y sets if λ is small. This justifies our claim for the large solutions in this case.

Similar arguments imply that when $f'(0) = f(0) = 0$, there is a unique small non-trivial positive solution for large λ and this holds uniformly in λ. Moreover, since $-\Delta u = f(u)$ has no non- constant non-negative bounded solution on R^2, we easily see by blowing up arguments similar to those in the proof of Theorem 3 in [4] that positive solutions of (1) for large λ have either sup norm small or large (and this is uniform in s).

Thus, to complete our proof, we need only study the behaviour of the large positive solutions of (1) for λ large in case (iiγ) (or case (iiμ)) or the positive solutions with $\|u\|_\infty$ near a in case (iiα). If $f(0) > 0$ or if $f'(0) > 0$ then, since the boundary of D_s changes quite smoothly in s, it is easy to see that subsolutions methods hold uniformly in s. In particular, in the second case above, the uniform estimates for the positive solution in the proof of Theorem 2 in [5] hold uniformly in s and the rest of the proof of Theorem 2 is readily seen to hold uniformly in s. In case (iiγ), it is easily seen that the only difficulty is to prove Step 2 of the proof of Lemma 1 (in [5]) holds uniformly in s. Note that w_0 depends on s but depends quite smoothly upon s. Here w_0^s is the solution of $-\Delta u = 1$ on D_s with Dirichlet boundary conditions. Suppose, as before, that y_n are positive solutions of (1) on D_{s_n} such that $\lambda_n \to \infty$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. The proof there shows that $-\Delta(\lambda_n^{-1}u_n - Cw_0^{s_n})$ tends to zero in $L^p(D_{s_n})$ and thus by $W^{2,p}$ estimates as in

[12] or [17] (including their remarks on the constants) $\|\lambda_n^{-1}u_n - Cw_0^{s_n}\|_{2,p,D_{s_n}} \to 0$. Here we mean the $W^{2,p}$ norm on D_{s_n}. Thus by the Sobolev embedding theorem (including the remarks in [1] or [17] on the dependence of the constants on the domain), $\lambda_n^{-1}u_n - Cw_0^{s_n}$ is small in C^1 on D_{s_n} uniformly in n. Hence our estimates in Lemma 1 in [5] are easily seen to be true if we prove there exist $C_1, C_2 > 0$ independent of n such that $C_1 d(x, \partial D_{s_n}) \le w_0^{s_n}(x) \le C_2 d(x, \partial D_{s_n})$ for all n. This follows easily by using barriers and by using solutions on balls as subsolutions. (Remember that D_s is quite smooth and quite smoothly dependent upon s.) The case where $p < 1$ is similar (that is, case (iiμ)). It remains to consider the case where $f(0) = f'(0) = 0$. Then, as we commented before, blow up arguments are easily seen to be valid uniformly in s and it is easy to combine the ideas in the previous paragraph with the ideas in the proof of Theorem 3 in [5] to show that argument there is valid uniformly in s. The proof of Theorem 4 in [5] can similarly be shown to be uniform in s. Indeed our special geometry enables us to simplify considerably the proof there. Thus we have proved the uniform in s behaviour at the extremities of W_s and the proof of Theorem 1 is complete.

REMARKS. 1. The uniform behaviour (in s) of the extremities of W_s can be proved rather more generally in many cases. Indeed, it is not difficult to generalize our theorem to arbitrary domains C^2 close to those in the theorem (for fixed f). If f satisfies (iiβ) and if either $f(0) > 0$ or $f(0)$ and $f'(0) > 0$, it can be shown that Theorem 1 continues to hold for domains close in the sense of [6] to a domain of type R. Here in the proof it is necessary to replace $C(B)$ by $L^p(B)$ for p large and prove closeness in L^p norms. This allows for much more general domains. Domain variation ideas imply that this result is not true for domains C^0 close if $f(0) = f'(0) = 0$.

2. It would be very interesting to prove results of this type in more than 2 dimensions. There are extra difficulties here because an example in [5] shows that, even for a ball, the connectedness is not so simple a problem. Moreover, critical Sobolev exponents now have an influence while it is very unclear how to generalize Proposition 1. However, our techniques do apply to domains C^2 close to a ball in R^n (where what is meant by close depends on f) provided that p in Assumpton (1) satisfies $p < (n-2)^{-1}(n+2)$, the q in Assumption (iiτ) satisfies $q < (n-2)^{-1}(n+2)$ and the equation $-\Delta u = f(u)$ has no bounded non constant positive solutions on R^n. Some conditions ensuring that the last condition holds can be found in the last section of [5]. As in Remark 1 we can allow perturbations of the balls of much more general type for asymptotically linear f's, with $f(0) > 0$ or with $f(0) = 0$ and $f'(0) > 0$.

3. To generalize our results to the Gelfand equation, we need better estimates for large solutions for small λ while to generalize our results to problems of Hess type (where we now try to prove that W_λ has the same number of components as when D is a ball), the difficulty is to prove good estimates for the number of positive solutions for large λ (especially the unstable solutions). It seems that this last problem is much more complicated and it is very unlikely that as simple a result is true in this case. However, it seems very likely that partial results can be obtained by our techniques.

REFERENCES

[1] R. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] H. AMANN, *Multiple fixed points of asymptotically linear mappings*, J. Funct. Anal. **17** (1974), 174–213.

[3] M. CRANDALL AND P. H. RABINOWITZ, *Bifurcation from simple eigenvalues*, J. Funct. Anal. **8** (1971), 321–340.

[4] E. N. DANCER, *The effect of domain shape on the number of positive solutions of certain nonlinear equations*, J. Differential Equations **74** (1988), 120–156.

[5] ———, *On the number of positive solutions of weakly nonlinear elliptic equations when a parameter is large*, Proc. London Math. Soc. **53** (1986), 429–452.

[6] ———, *The effect of domain shape on the number of positive solutions of certain nonlinear equations II*, J. Differential Equations **87** (1990), 316–339.

[7] ———, *A note on bifurcation from infinity*, Quart. J. Math. Oxford **25** (1974), 81–84.

[8] ———, *Weakly nonlinear Dirichlet problems on long or thin domains*, Memoirs Amer. Math. Soc. (to appear).

[9] ———, *Multiple fixed points of positive mappings*, J. Reine Angew. Math. **37** (1986), 46–66.

[10] ———, *A note on the uniqueness for the plasma problem*, Z. Angew. Math. Phys. **43** (1992), 397–400.

[11] B. GIDAS, W. NI AND L. NIRENBERG, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[12] D. GILBARG AND N. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, second edition, Springer, Berlin, 1983.

[13] R. HARDT AND L. SIMON, *Nodal sets for solutions of elliptic equations*, J. Differential. Geom. **30** (1989), 505–522.

[14] P. HARTMAN AND A. WINTNER, *On the local behaviour of solutions of non-parabolic partial differential equations, III, Approximation by spherical harmonics*, Amer. J. Math. **77** (1955), 453–474.

[15] P. HESS, *On multiple solutions of nonlinear elliptic eigenvalue problems*, Comm. Partial Differential Equations **6** (1981), 951–961.

[16] H. KIELHOFER, *Smoothness of global positive branches of nonlinear elliptic problems over symmetric domains*, Math. Z. (to appear).

[17] O. LADYZHENSKAYA AND N. URAL'TSEVA, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[18] A. MELAS, *On the nodal line of the second eigenfunction of the Laplacian in R^2*, J. Differential Geom. **35** (1992), 255–263.

[19] J. SAUT AND R. TEMAN, *Generic properties of nonlinear boundary value problems*, Comm. Partial Differential Equations 4 (1979), 293–319.

[20] G. WHYBURN, *Topological Analysis*, Princeton Univ. Press, Princeton, 1958.

Manuscript received April 20, 1993

E. N. DANCER
School of Mathematics and Statistics
University of Sidney
Sydney, NSW 2006, AUSTRALIA

E-mail address: dancer_n@maths.su.oz.au