SOLUTION SETS OF BOUNDARY VALUE PROBLEMS FOR NONCONVEX DIFFERENTIAL INCLUSIONS

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(Submitted by L. Górniewicz)

Dedicated to the memory of Karol Borsuk

1. Introduction and preliminaries

Topological properties of the solution set of Cauchy problems for differential inclusions have been investigated by several authors [16], [24], [14], [23], [10], [19], [3], [15]. Less attention has been, so far, devoted to analogous questions for boundary value problems.

In the present paper we consider boundary value problems of the type

\[
\begin{aligned}
& x''(t) \in F(t, x(t), x'(t)), \\
& x(0) = x(1) = 0,
\end{aligned}
\]

(BV)

where \( F \) is a multifunction from \( I \times \mathbb{R}^q \times \mathbb{R}^q, I = [0, 1], \) to the non-empty compact subsets of \( \mathbb{R}^q \). If \( F \) is Lipschitzean, we prove that the solution set \( S_F \) of (BV) is a retract of the Sobolev space \( W^{2,1}(I, \mathbb{R}^q) \). In particular, \( S_F \) is contractible and hence arcwise connected. Whenever \( F \) is convex valued and Lipschitzean, \( S_F \) is a retract also of \( C^1(I, \mathbb{R}^q) \). Finally, in the nonconvex case, under a continuity assumption on \( F \), it is proved that \( S_F \) is non-empty.

To establish the retraction property of \( S_F \), when \( F \) is Lipschitzean, we use some recent results due to Ricceri [21] and Bressan, Cellina and Fryszkowski [4].
who have studied the existence of a retraction of a Banach space $X$ onto the set of the fixed points of a contractive multifunction from $X$ into itself. Developments and applications of such ideas can be found in Rybiński [22]. The nonemptiness of $S_F$, when $F$ is continuous, is obtained as in Papageorgiou [18], by a technique based on a selection theorem for decomposable valued multifunctions of Antosiewicz-Cellina type [1], [9], [5].

Unlike the nonconvex case, boundary value problems of the type (BV) with $F$ compact convex valued have been studied by many authors. We mention, among others, Pruszkó [20], also for a historical outline and an extensive list of references, and Erbe and Krawcewicz [7] and Frigon [8], who use an approach based on the topological transversality method of Granas, Guenther and Lee [11].

Let $X$ be a metric space with distance $d_X$. For $x \in X$ and $A$ a non-empty subset of $X$, we set $d_X(x, A) = \inf_{a \in A} d_X(x, a)$. We denote by $K(X)$ the space of all non-empty closed bounded subsets of $X$ equipped with the Hausdorff metric

$$D_X(A, B) = \max \left\{ \sup_{b \in B} d_X(b, A), \sup_{a \in A} d_X(a, B) \right\}, \quad A, B \in K(X).$$

Moreover $C(X)$, where $X$ is a normed space, denotes the space of all non-empty, convex, closed, bounded subsets of $X$ endowed with the Hausdorff metric $D_X$. By $B_X(x, r)$ (resp. $\bar{B}_X(x, r)$) we mean an open (resp. closed) ball in $X$ with center $x \in X$ and radius $r > 0$ (resp. $r \geq 0$).

Let $X, Y$ be metric spaces. A multifunction $F : X \to K(Y)$ is said to be Hausdorff lower (resp. upper) semicontinuous if, for every $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $F(x_0) \subset \{ y \in Y \mid d_Y(y, F(x)) < \varepsilon \}$ (resp. $F(x) \subset \{ y \in Y \mid d_Y(y, F(x_0)) < \varepsilon \}$) for every $x \in B_X(x_0, \delta)$. $F$ is called Hausdorff continuous if $F$ is Hausdorff lower and upper semicontinuous. A multifunction $F : T \to K(Y)$, $T$ and interval of $\mathbb{R}$, is said to be measurable if for every closed subset $C$ of $Y$, the set $\{ t \in T \mid F(t) \cap C \}$ is Lebesgue measurable. We refer to Castaing and Valadier [6] for further properties of measurable multifunctions.

To study problem (BV) we introduce the following assumptions about $F$.

Let $F : I \times \mathbb{R}^q \times \mathbb{R}^q \to K(\mathbb{R}^q)$, $I = [0, 1]$, be a multifunction.

We say that $F$ satisfies (L) if:

(i) For every $(x, y) \in \mathbb{R}^q \times \mathbb{R}^q$ the multifunction $t \to F(t, x, y)$ is measurable and satisfies

$$D_{\mathbb{R}^q}(F(t, 0, 0), \{0\}) \leq m(t) \quad \text{for} \quad t \in I,$$

where $m : I \to \mathbb{R}$ is non-negative and integrable.

(ii) For every $(t, x_1, y_1), (t, x_2, y_2) \in I \times \mathbb{R}^q \times \mathbb{R}^q$ we have

$$D_{\mathbb{R}^q}(F(t, x_1, y_1), F(t, x_2, y_2)) \leq a|x_1 - y_1| + b|x_2 - y_2|,$$
where \( a \geq 0, b \geq 0 \) and \( a + b = k < 1 \).

We say that \( F \) satisfies (C) if:

(i) For every \((x, y) \in \mathbb{R}^a \times \mathbb{R}^b\) the multifunction \( t \rightarrow F(t, x, y) \) is measurable and satisfies

\[
D_{\mathbb{R}^a}(F(t, x, y), \{0\}) \leq m(t), \quad \text{for every } (t, x, y) \in I \times \mathbb{R}^a \times \mathbb{R}^b,
\]

where \( m : I \rightarrow \mathbb{R} \) is non-negative and square integrable.

(ii) For \( t \in I \) a.e. the multifunction \((x, y) \rightarrow F(t, x, y)\) is Hausdorff continuous.

Suppose that \( F \) satisfies (L) or (C). A function \( x : I \rightarrow \mathbb{R}^a \) is said to be a solution of the boundary value problem (BV) if: (i) \( x \) is absolutely continuous with \( x(0) = x(1) = 0 \), (ii) \( x' \) is absolutely continuous, and (iii) \( x''(t) \in F(t, x(t), x'(t)) \), \( t \in I \) a.e. The set of all solutions of (BV) is called the solution set of (BV) and denoted by \( S_F \).

The above definition of a solution remains valid when \( F \) is, in particular, single valued.

Let \( I = [0, 1] \). We denote by \( C(I, \mathbb{R}^a) \) (resp. \( C^1(I, \mathbb{R}^a) \)) the Banach space of all continuous (resp. continuously differentiable functions) \( x : I \rightarrow \mathbb{R}^a \) endowed with the norm

\[
\|x\|_C = \max_{t \in I} |x(t)| \quad \text{(resp. } \|x\|_{C^1} = \max\{\|x\|_C, \|x'\|_C\})).
\]

As usual, \( L^1(I, \mathbb{R}^a) \) (resp. \( L^2(I, \mathbb{R}^a) \)) is the Banach space of all (equivalence classes of) integrable (resp. square integrable) functions \( x : I \rightarrow \mathbb{R}^a \) equipped with the norm \( \|x\|_{L^1} = \int_I |x(t)| dt \) (resp. \( \|x\|_{L^2} = \int_I |x(t)|^2 dt \)). Furthermore, \( W^{2,1}(I, \mathbb{R}^a) \) denotes the Sobolev space of all functions \( x : I \rightarrow \mathbb{R}^a \) such that \( x \) and \( x' \) are absolutely continuous (thus, with \( x'' \in L^1(I, \mathbb{R}^a) \)), endowed with the norm

\[
\|x\|_{W^{2,1}} = \|x\|_{L^1} + \|x'\|_{L^1} + \|x''\|_{L^1}.
\]

We recall that a set \( K \subset L^1(I, \mathbb{R}^a) \) is said to be decomposable (see Hiai and Umegaki [12]) if \( u \chi_J + v \chi_I, J \in K \) whenever \( u, v \in K \) and \( J \) is any measurable subset of \( I \). Here \( \chi_A \) stands for the characteristic function of a set \( A \subset I \). The family of all non-empty, decomposable, closed, bounded subsets of \( L^1(I, \mathbb{R}^a) \) is denoted by \( D_{L^1}(I, \mathbb{R}^a) \).

Let \( Z \) be a Hausdorff topological space. A subspace \( X \) of \( Z \) is said to be a retract of \( Z \) if there is a continuous map \( r : Z \rightarrow X \) satisfying \( r(x) = x \) for every \( x \in X \). Any such map \( r \) is called retraction of \( Z \) onto \( X \). Clearly, if \( X \) is a retract of \( Z \), then \( X \) is closed in \( Z \). A metrizable space \( X \) is said to be an absolute retract (for metrizable spaces) if for every homeomorphism \( h \) mapping \( X \) onto a closed subset \( h(X) \) of a metrizable space \( Y \), the set \( h(X) \) is a retract of \( Y \). We recall that
every retract of a convex set of a normed space is an absolute retract (see Borsuk [2], p. 85).

2. Topological properties of $S_F$

**Theorem 1.** Let $F : I \times R^q \times R^q \to \mathcal{K}(R^q)$ satisfy (L). Then the solution set $S_F$ of the boundary value problem (BV) is a retract of $W^{2,1}(I, R^q)$.

**Proof.** For $u \in L^1(I, R^q)$ we denote by $x(u) : I \to R^q$ the solution of the boundary value problem

$$
\begin{align*}
(P_u) & \quad \begin{cases} 
  x''(t) = u(t) \\
  x(0) = x(1) = 0.
\end{cases}
\end{align*}
$$

This solution exists, is unique, and is given by

$$
(2.1) \quad x(u)(t) = \int_0^t \left( \int_0^\tau su(s) \, ds - \int_0^1 (1 - s)u(s) \, ds \right) \, d\tau, \quad t \in I.
$$

For $u \in L^1(I, R^q)$, set

$$
(2.2) \quad \mathcal{U}(u) = \{ \sigma \in L^1(I, R^1) \mid \sigma(t) \in F(t, x(u)(t), x'(u)(t)), \ t \in I \text{ a.e.} \}.
$$

Clearly $\mathcal{U}(u)$ is a non-empty decomposable closed subset of $L^1(I, R^q)$. From $F(t, x(u)(t), x'(u)(t)) \subseteq F(t, 0, 0) + \mathcal{B}_{R^q}(0, D_{R^q}^n(F(t, x(u)(t), x'(u)(t)), F(t, 0, 0)))$ and assumption (L), it follows that $\mathcal{U}(u)$ is bounded in $L^1(I, R^q)$. Thus (2.2) defines a multifunction $\mathcal{U} : L^1(I, R^q) \to D_{L^1(I, R^q)}$.

For every $u_1, u_2 \in L^1(I, R^q)$ we have

$$
(2.3) \quad D_{L^1}(\mathcal{U}(u_1), \mathcal{U}(u_2)) \leq k \| u_1 - u_2 \|_{L^1},
$$

where $k$ is the constant occurring in (L). Indeed, let $u_1, u_2 \in L^1(I, R^q)$. Let $x(u_1)$ and $x(u_2)$ be the solutions of $(P_{u_1})$ and $(P_{u_2})$, respectively. From (2.1) we have

$$
(2.4) \quad \| x(u_1) - x(u_2) \|_C \leq \| u_1 - u_2 \|_{L^1}, \quad \| x'(u_1) - x'(u_2) \|_C \leq \| u_1 - u_2 \|_{L^1}.
$$

Let $\sigma_1 \in \mathcal{U}(u_1)$ be arbitrary. Since the multifunction $\Phi : I \to \mathcal{K}(R^q)$ given by

$$
\Phi(t) = F(t, x(u_2)(t), x'(u_2)(t)) \cap \mathcal{B}_{R^q}(\sigma_1(t), d_{R^q}(\sigma_1(t), F(t, x(u_2)(t), x'(u_2)(t))),
$$

for $t \in I$ is measurable, there exists $\sigma_2 \in \mathcal{U}(u_2)$ satisfying

$$
(2.5) \quad |\sigma_1(t) - \sigma_2(t)| = d_{R^q}(\sigma_1(t), F(t, x(u_2)(t), x'(u_2)(t))), \quad t \in I \text{ a.e.}
$$
By virtue of (2.5), assumption (L) (ii), and (2.4) we have:

\[\|\sigma_1 - \sigma_2\| = \int_I d_{\mathbb{R}^1}(\sigma_1(t), F(t, x(u_2)(t), x'(u_2)(t)))dt\]
\[\leq \int_I d_{\mathbb{R}^1}(F(t, x(u_1)(t), x'(u_1)(t)), F(t, x(u_2)(t), x'(u_2)(t)))dt\]
\[\leq \int_I (a|x(u_1)(t) - x(u_2)(t)| + b|x'(u_1)(t) - x'(u_2)(t)|)dt\]
\[\leq k\|u_1 - u_2\|_{L^1}\]

Hence \(d_{L^1}(\sigma_1, \mathcal{U}(u_2)) \leq k\|u_1 - u_2\|_{L^1}\) and thus, as \(\sigma_1 \in \mathcal{U}(u_1)\) is arbitrary,

\[\sup_{\sigma_1 \in \mathcal{U}(u_1)} d_{L^1}(\sigma_1, \mathcal{U}(u_2)) \leq k\|u_1 - u_2\|_{L^1}.

Combining this with the analogous inequality obtained by interchanging the roles of \(u_1\) and \(u_2\) gives (2.3).

Put \(\text{Fix}(\mathcal{U}) = \{u \in L^1(I, \mathbb{R}^q) \mid u \in \mathcal{U}(u)\}\). By a result of Nadler [17], \(\text{Fix}(\mathcal{U})\) is a non-empty closed subset of \(L^1(I, \mathbb{R}^q)\). By a theorem of Bressan, Cellina and Fryszkowski [4] the set \(\text{Fix}(\mathcal{U})\) is a retract of \(L^1(I, \mathbb{R}^q)\). Hence there exists a continuous map \(r : L^1(I, \mathbb{R}^q) \rightarrow \text{Fix}(\mathcal{U})\) satisfying \(r(u) = u\) for every \(u \in \text{Fix}(\mathcal{U})\). For \(x \in W^{2,1}(I, \mathbb{R}^q)\) define \(Rx : I \rightarrow \mathbb{R}^q\) by

\[(2.6)\quad (Rx)(t) = \int_0^t \left(\int_0^s r(x''(s)) ds - \int_t^1 (1 - s)r(x''(s)) ds\right) dr, \quad t \in I.

Clearly, \(Rx\) coincides with the solution of the boundary value problem

\[
\begin{align*}
   y''(t) &= r(x'')(t) \\
   y(0) &= y(1) = 0.
\end{align*}
\]

As \(r(x'') \in \text{Fix}(\mathcal{U})\), we have \(r(x'') \in \mathcal{U}(r(x''))\) and thus

\[(Rx)'(t) = r(x')(t) \in F(t, (Rx)(t), (Rx)'(t)), \quad t \in I \text{ a.e.}\]

Since, in addition, \(Rx\) and \((Rx)'\) are absolutely continuous and \((Rx)(0) = (Rx)(1) = 0\), it follows that \(Rx \in S_F\). Thus, denoting by \(R\) the map which associates with each \(x \in W^{2,1}(I, \mathbb{R}^q)\) the function \(Rx\) given by (2.6), we have:

\[R : W^{2,1}(I, \mathbb{R}^q) \rightarrow S_F.\]

The map \(R\) is continuous. In fact, let \(x_0, x \in W^{2,1}(I, \mathbb{R}^q)\) and \(\varepsilon > 0\) be arbitrary. From (2.6), by simple calculations, we have

\[\|Rx - Rx_0\|_{W^{2,1}} \leq 3\|r(x'') - r(x_0'')\|_{L^1}.\]

Take \(\delta > 0\) so that \(\|r(u) - r(x_0'')\|_{L^1} < \varepsilon / 3\) for every \(u \in B_{L^1}(x_0'', \delta)\). Let \(x \in B_{W^{2,1}}(x_0, \delta)\) be arbitrary. As \(x'' \in B_{L^1}(x_0'', \delta)\) we have \(\|r(x'') - r(x_0'')\|_{L^1} < \varepsilon / 3\), and thus \(\|Rx - Rx_0\|_{W^{2,1}} < \varepsilon\). Hence \(R\) is continuous.
For each $x \in S_F$ we have $Rx = x$. Indeed; let $x \in S_F$ be arbitrary. Put $u = x''$. Denoting by $y(u)$ the solution of $(P_u)$ we have $y(u) = x$, and so $u(t) = x''(t) \in F(t, x(t), x'(t)) = F(t, y(u)(t), y'(u)(t))$, $t \in I$ a.e. Hence $u \in U(u)$, which implies $r(u) = u$ and thus, $r(x'') = x''$. Consequently, for each $t \in I$,

$$
(Rx)(t) = \int_0^t \left( \int_0^s sr(x'')(s) \, ds - \int_0^1 (1 - s)r(x'')(s) \, ds \right) \, d\tau
= \int_0^t \left( \int_0^s sz''(s) \, ds - \int_0^1 (1 - s)x''(s) \, ds \right) \, d\tau = x(t),
$$

that is $Rx = x$. It follows that $R$ is a retraction of $W^{2,1}(I, \mathbb{R}^q)$ onto $S_F$. This completes the proof.

3. Continuation

**Theorem 2.** Let $F : I \times \mathbb{R}^q \times \mathbb{R}^q \to C(\mathbb{R}^q)$ satisfy (L), where the function $m : I \to \mathbb{R}$ is square integrable. Then the solution set $S_F$ of the boundary value problem (BV) is a retract of $C^1(I, \mathbb{R}^q)$.

**Proof.** For $y \in C^1(I, \mathbb{R}^q)$ we set

$$
(3.1) \quad U(y) = \{u \in L^1(I, \mathbb{R}^q) \mid u(t) \in F(t, y(t), y'(t)), \quad t \in I \text{ a.e.}\}.
$$

Clearly $U(y)$ is a non-empty, convex, closed and bounded subset of $L^1(I, \mathbb{R}^q)$. For $y \in C^1(I, \mathbb{R}^q)$ we define

$$
(3.2) \quad F(y) = \{x(u) \mid u \in U(y)\}.
$$

Here, for $u \in U(y)$, $x(u)$ denotes the solution of $(P_u)$.

$F(y)$ is a non-empty, convex and compact subset of $C^1(I, \mathbb{R}^q)$. It is evident that $F(y)$ is non-empty and convex. To show that $F(y)$ is compact, consider an arbitrary sequence $\{z_n\} \subset F(y)$. Let $\{u_n\} \subset U(y)$ be such that $z_n = x(u_n), n \in \mathbb{N}$. Since, for $t \in I$ a.e.,

$$
u_n(t) \in F(t, 0, 0) + \tilde{B}_{R^q}(0, D_{R^q}(F(t, 0, 0), F(t, y(t), y'(t)))) \subset \tilde{B}_{R^q}(0, m(t) + a|y(t)| + b|y'(t)|),
$$

where the function $t \to m(t) + a|y(t)| + b|y'(t)|$ is square integrable, there exists a subsequence, say $\{u_n\}$, which converges weakly in $L^2(I, \mathbb{R}^q)$ to some $u \in L^2(I, \mathbb{R}^q)$. Clearly, $u \in L^1(I, \mathbb{R}^q)$ and $\{u_n\}$ converges to $u$ weakly in $L^1(I, \mathbb{R}^q)$. By Mazur's
theorem [13] it is easy to see that \( u \in U(y) \). Now, for \( n \in \mathbb{N} \) and \( t \in I \), we have:

\[
(3.3) \quad x(u_n)(t) - x(u)(t) = \int_0^t \left( \int_0^r s(u_n(s) - u(s)) \, ds \right) \, dt \\
- \int_0^t \left( \int_0^1 (1 - s)(u_n(s) - u(s)) \, ds \right) \, dt
\]

\[
(3.4) \quad x'(u_n)(t) - x'(u)(t) = \int_0^t s(u_n(s) - u(s)) \, ds \\
- \int_t^1 (1 - s)(u_n(s) - u(s)) \, ds.
\]

Since \( \{u_n\} \) converges to \( u \) weakly in \( L^1(I, \mathbb{R}^q) \), from (3.3) and (3.4) it follows that \( \{x(u_n)\} \) and \( \{x'(u_n)\} \) converge in \( C(I, \mathbb{R}^q) \) to \( x(u) \) and \( x'(u) \), respectively. Hence \( \{x(u_n)\} \) converges to \( x(u) \) in \( C^1(I, \mathbb{R}^q) \). As \( x(u) \in F(y) \), the set \( F(y) \) is compact. Thus (3.2) defines a non-empty, convex, compact valued multifunction

\[
F : C^1(I, \mathbb{R}^q) \to C(C^1(I, \mathbb{R}^q)).
\]

For every \( y_1, y_2 \in C^1(I, \mathbb{R}^q) \) we have

\[
(3.5) \quad D_{C^1}(F(y_1), F(y_2)) \leq k \|y_1 - y_2\|_{C^1},
\]

where \( k \) is the constant occurring in (L). Indeed, let \( y_1, y_2 \in C^1(I, \mathbb{R}^q) \). Let \( z_1 \in F(y_1) \) be arbitrary, thus \( z_1 = x(u_1) \) for some \( u_1 \in U(y_1) \). As in the proof of Theorem 1, take \( u_2 \in U(y_2) \) satisfying

\[
(3.6) \quad |u_1(t) - u_2(t)| = d_{\mathbb{R}^q}(u_1(t), F(t, y_2(t), y'_2(t))), \quad t \in I \text{ e.e.},
\]

and set \( z_2 = x(u_2) \). Clearly, \( z_2 \in F(y_2) \). Using the representation of \( x(u_1) \) and \( x(u_2) \) given by (2.1), by simple calculations, for every \( t \in I \), we have:

\[
|x(u_1)(t) - x(u_2)(t)| \\
= |(t - 1) \int_0^t s(u_1(s) - u_2(s)) \, ds - t \int_t^1 (1 - s)(u_1(s) - u_2(s)) \, ds| \\
\leq \int_0^t |u_1(s) - u_2(s)| \, ds + \int_t^1 |u_1(s) - u_2(s)| \, ds = \int_I |u_1(s) - u_2(s)| \, ds.
\]

From this, using (3.6) and assumption (L) (ii), for every \( t \in I \) we obtain:

\[
|x(u_1)(t) - x(u_2)(t)| \leq \int_I d_{\mathbb{R}^q}(u_1(t), F(t, y_2(t), y'_2(t))) \, dt \\
\leq \int_I D_{\mathbb{R}^q}(F(t, y_1(t), y'_1(t)), F(t, y_2(t), y'_2(t))) \, dt \\
\leq \int_I (a|y_1(t) - y_2(t)| + b|y'_1(t) - y'_2(t)|) \, dt \\
\leq k \|y_1 - y_2\|_{C^1}.
\]
Consequently $\|z_1 - z_2\|_{C^1} \leq k\|y_1 - y_2\|_{C^1}$. Likewise one can show that $\|z'_1 - z'_2\|_{C^1} \leq k\|y'_1 - y'_2\|_{C^1}$. Hence, $\|z_1 - z_2\|_{C^1} \leq k\|y_1 - y_2\|_{C^1}$. A fortiori, $d_{C^1}(z_1, \mathcal{F}(y_2)) \leq k\|y_1 - y_2\|_{C^1}$, and thus, as $z_1 \in \mathcal{F}(y_1)$ is arbitrary,

$$\sup_{z_1 \in \mathcal{F}(y_1)} d_{C^1}(z_1, \mathcal{F}(y_2)) \leq k\|y_1 - y_2\|_{C^1}.$$ 

From this and the analogous inequality obtained by interchanging the roles of $y_1$ and $y_2$ we obtain (3.5).

Put $\text{Fix}(\mathcal{F}) = \{y \in C^1(I, \mathbb{R}^q) \mid y \in \mathcal{F}(y)\}$, and observe that $\text{Fix}(\mathcal{F})$ is a non-empty closed subset of $C^1(I, \mathbb{R}^q)$. By a result of Ricceri [21] $\mathcal{F}(y)$ is a retract of $C^1(I, \mathbb{R}^q)$. It is routine to show that $\text{Fix}(\mathcal{F}) = S_F$. Hence $S_F$ is a retract of $C^1(I, \mathbb{R}^q)$ and the proof of the theorem is complete.

**Remark 1.** By Theorem 1 (resp. Theorem 2), the space $S_F$ with the $W^{2,1}(I, \mathbb{R}^q)$ (resp. $C^1(I, \mathbb{R}^q)$) metric is an absolute retract.

**Remark 2.** Theorem 2 is no longer true if $F$ is not convex valued. To see this, denote by $S$ the solution set of the boundary value problem

$$\begin{align*}
\left\{ \begin{array}{l}
x''(t) \in \{-1, 1\}, \\
x(0) = x(1) = 0.
\end{array} \right.
\end{align*}$$

(3.7)

Since $S$ is not closed in $C^1(I, \mathbb{R})$, the set $S$ cannot be a retract of $C^1(I, \mathbb{R})$. On the other hand, from Theorem 1, $S$ is a retract of $W^{2,1}(I, \mathbb{R})$ and so $S$ is closed in $W^{2,1}(I, \mathbb{R})$.

4. An existence result

**Theorem 3.** Let $F : I \times \mathbb{R}^q \times \mathbb{R}^q \to \mathcal{K}(\mathbb{R}^q)$ satisfy (C). Then the solution set $S_F$ of the boundary value problem (BV) is non-empty.

**Proof.** For $u \in L^1(I, \mathbb{R}^q)$ denote by $y(u)$ the solution of $(P_u)$. Set

$$\Omega = \{y \in C^1(I, \mathbb{R}^q) \mid y = y(u) \text{ for some measurable } u \text{ with } |u(t)| \leq m(t), \, t \in I \text{ a.e.} \}.$$ 

Clearly $\Omega$ is non-empty and convex. Moreover $\Omega$, endowed with the $C^1(I, \mathbb{R}^q)$ metric, is a compact space. To see this, let $\{y(u_n)\} \subset \Omega$ be an arbitrary sequence where, for each $n \in \mathbb{N}$, $u_n : I \to \mathbb{R}^q$ is measurable and $|u_n(t)| \leq m(t), \, t \in I \text{ a.e.}$ As $m$ is square integrable, there is a subsequence, say $\{u_n\}$, which converges weakly to some $u$ in $L^2(I, \mathbb{R}^q)$ and so also in $L^1(I, \mathbb{R}^q)$. By the Mazur theorem [13] one
has \(|u(t)| \leq m(t), \ t \in I \text{ a.e.}\), and so \(y(u) \in \Omega\). By using the representation of the solution of \((P_u)\) furnished by (2.1), it follows that \(\{y(u_n)\}\) converges to \(y(u)\) in \(C^1(I, \mathbb{R}^q)\), proving that \(\Omega\) is compact.

For \(y \in \Omega\), let \(\mathcal{U}(y)\) be given by (3.1). As \(\mathcal{U}(y)\) is a non-empty, decomposable, closed, bounded subset of \(L^1(I, \mathbb{R}^q)\), (3.1) defines a multifunction \(\mathcal{U}: \Omega \to \mathcal{D}_{L^1(I, \mathbb{R}^q)}\). It is routine to verify that \(\mathcal{U}\) is Hausdorff lower semicontinuous. By virtue of Theorem 3 of Bressan and Colombo [5], there exists a continuous function \(\sigma: \Omega \to L^1(I, \mathbb{R}^q)\) satisfying

\[
(4.1) \quad \sigma(y) \in \mathcal{U}(y) \quad \text{for every } y \in \Omega.
\]

For \(y \in \Omega\), let \(x(y): I \to \mathbb{R}^q\) denote the solution of the boundary value problem

\[
\begin{align*}
  x''(t) &= \sigma(y)(t) \\
  x(0) &= x(1) = 0.
\end{align*}
\]

This solution exists, is unique, and is given by

\[
(4.2) \quad x(y)(t) = \int_0^t \left( \int_0^r s \sigma(y)(s) \, ds - \int_r^1 (1 - s) \sigma(y)(s) \, ds \right) \, dr, \quad t \in I.
\]

Clearly \(x(y) \in \Omega\). Denote by \(T: \Omega \to \Omega\) the map defined by \(Ty = x(y), \ y \in \Omega\). \(T\) is continuous. Indeed, let \(y_0, y \in \Omega\). From (4.2), we have

\[
\|Ty - Ty_0\|_C \leq \|\sigma(y) - \sigma(y_0)\|_{L^1},
\]

\[
\| (Ty)' - (Ty_0)' \|_C \leq \| \sigma(y) - \sigma(y_0) \|_{L^1}.
\]

Hence

\[
\|Ty - Ty_0\|_{C^1} \leq \|\sigma(y) - \sigma(y_0)\|_{L^1},
\]

which implies that \(T\) is continuous, for \(\sigma: \Omega \to L^1(I, \mathbb{R}^q)\), is so. By Schauder’s fixed point theorem, there exists \(y \in \Omega\) such that \(y = Ty\), thus

\[
y(t) = \int_0^t \left( \int_0^r s \sigma(y)(s) \, ds - \int_r^1 (1 - s) \sigma(y)(s) \, ds \right) \, dr, \quad t \in I.
\]

Since \(y\) and \(y'\) are absolutely continuous, \(y(0) = y(1) = 0\) and, by virtue of (4.1) and (3.1),

\[
y''(t) = \sigma(y)(t) \in F(t, y(t), y'(t)), \quad t \in I \text{ a.e.,}
\]

it follows that \(y\) is a solution of the boundary value problem \((BV)\). Thus \(S_F\) is non-empty, completing the proof.
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