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ON THE SOLVABILITY OF A TWO POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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(Submitted by Ky Fan)

Dedicated to the memory of Karol Borsuk

1. Introduction

We consider the boundary value problem

$$u'' + u + g(x,u) = h(x) \qquad \text{in } (0,\pi),\ u(0) = u(\pi) = 0, \tag{1.1}$$

where $h \in L^1(0,\pi)$ is given and $g : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $g(x,u)$ is continuous in $u \in \mathbb{R}$ for a.e. $x \in (0,\pi)$ and is measurable in $x \in (0,\pi)$ for all $u \in \mathbb{R}$. We assume throughout this paper that

(H1) For each $r > 0$, there exists $\alpha_r \in L^1(0,\pi)$ such that

$$|g(x,u)| \le \alpha_r(x)$$

for a.e. $x \in (0,\pi)$ and $|u| \le r$;

(H2) There exists $\Gamma \in L^1(0,\pi)$ such that

$$0 \le \limsup_{|u|\to\infty} \frac{g(x,u)}{u} \le \Gamma(x) \tag{1.2}$$

uniformly for a.e. $x \in (0,\pi)$ and

$$\|\Gamma\|_{L^1} < 1. \tag{1.3}$$

The solvability of the problem (1.1) has been extensively studied in literature if Γ is assumed to be bounded. Existence theorems for a solution to (1.1) are obtained if $\Gamma(X) \le 3$ for a.e. $x \in (0,\pi)$, with the strict inequality holding on a subset of $(0,\pi)$ of positive measure (see [1, 6]). A further result along the line is proved in [4] by assuming that for some $\rho > 1$ and a small $\varepsilon > 0$,

$$\limsup_{u\to\infty} \frac{g(x,u)}{u} \le \rho - 1 - \varepsilon,$$
$$\limsup_{u\to\infty} \frac{g(x,u)}{u} \le \rho(\rho^{\frac{1}{2}} - 1)^{-2} - 1 - \varepsilon.$$

The purpose of this paper is to obtain solvability results for (1.1) when Γ is a function in $L^1(0,\pi)$ satisfying (1.3). Thus our results can be applied to the problem (1.1) when the non-linear term g is given, for example, by

$$g(x,u) = \frac{u(1-e^{-|u|})}{(4x^{\frac{1}{2}})}$$

which cannot be covered by those in [1, 4, 6]. Our main result is Theorem 2 in §2, which is an existence theorem for a solution to (1.1) by assuming (1.3) and a Landesman-Lazer condition (see (2.7) below) originally obtained in [7]. In §3 we give in Theorem 3 solvability conditions for (1.1) in the absence of a Landesman-Lazer condition, which improves the main result in [5], where it is assumed that $\|\Gamma\|_{L^1} \le 1/15.87$. We note that the first inequality in (1.1) can be implied by the other conditions in the theorems (see (2.6) and (3.1) below). To prove our results using a Lyapunov type inequality shown in Lemma 1 and the well-known Leray-Schauder continuation method (see [3], Chap. 2).

In what follows we shall make use of the real Banach spaces $L^p(0,\pi)$, $C[0,\pi]$, $C^1[0,\pi]$ and the Sobolev spaces $H_0^1(0,\pi)$ and $W^{2,1}(0,\pi)$. The norms of $L^1(0,\pi)$ and $C[0,\pi]$ are denoted by $\|u\|_{L^1}$ and $\|u\|_C$, respectively. We recall that $W^{2,1}(0,\pi)$ is imbedded into $C^1[0,\pi]$ and $H_0^1(0,\pi)$ is compactly imbedded into $C[0,\pi]$ (see [2], Chap. 8). By a solution of (1.1), we mean a function $u \in H_0^1(0,\pi)$ solving the differential equation in (1.1) in the sense of distribution. It follows from standard regularity arguments that $u \in W^{2,1}(0,\pi)$ and satisfies the differential equation in (1.1) a.e. on $(0,\pi)$.

2. The Main Result

We recall from the theory of linear boundary value problems for second order differential equations that the eigenvalue problem

$$u'' + \lambda u = 0 \quad \text{in} \quad (0,\pi), \quad u(0) = u(\pi) = 0 \tag{2.1}$$

has simple eigenvalues at $\lambda = k^2$, with the corresponding eigenfunctions $\sin kx$ for $k = 1, 2, \ldots$. For $h \in L^1(0,\pi)$ the problem

$$u'' + u = h(x) \quad \text{in} \quad (0,\pi), \quad u(0) = u(\pi) = 0 \tag{2.2}$$

has a solution if and only if $\int_0^\pi h(x) \sin x \, dx = 0$. In this case, there exists a unique solution u to (2.2) such that $\int_0^\pi u(x) \sin x \, dx = 0$. More precisely, we have

$$u(x) = \int_0^\pi G(x,\xi)h(\xi)\, d\xi,$$

where

$$G(x,\xi) = (\sin x)(\xi \cos \xi)/\pi - \begin{cases} \cos x \sin \xi & \text{if } 0 \le \xi \le x, \\ \sin x \cos \xi & \text{if } x \le \xi \le \pi \end{cases}$$

is the Green function for the problem (2.1) when $\lambda = 1$. It is easy to see that

$$\max_{0 \le x,\xi \le \pi} |G(x,\xi)| = 1. \tag{2.3}$$

Now let $\lambda \in \mathbb{R}$ which is not an eigenvalue of (2.1). Then for any $h \in L^1(0,\pi)$ there exists a unique solution u to the problem

$$u'' + \lambda u = h(x) \quad \text{in } (0,\pi), \quad u(0) = u(\pi) = 0 \tag{2.4}$$

denoted by $u = K_\lambda h$. Moreover, $K_\lambda : L^1(0,\pi) \to H_0^1(0,\pi)$ is a continuous linear operator and hence, by the compact imbedding of $H_0^1(0,\pi)$ into $C[0,\pi]$, we infer that $K_\lambda : L^1(0,\pi) \to C[0,\pi]$ is a compact linear operator.

LEMMA 1. *Let $m \in L^1(0,\pi)$, $m(x) \ge 0$ for a.e. $x \in (0,\pi)$. If the problem*

$$w'' + (1 + m(x))w = 0 \quad \textit{in } (0,\pi), \ w(0) = w(\pi) = 0$$

has a non-trivial solution w, then either $w = \beta \sin x$ for some $\beta \in \mathbb{R} \setminus \{0\}$ or $\|m\|_{L^1} \ge 1$.

PROOF. It follows from the discussion above that

$$w(x) = (2/\pi)\left(\int_0^\pi w(\xi) \sin \xi \, d\xi\right) \sin x + \int_0^\pi G(x,\xi)m(\xi)w(\xi)\, d\xi. \tag{2.5}$$

Since $\int_0^\pi m(x)w(x)\sin x\,dx = 0$, taking the inner product in $L^2(0,\pi)$ of (2.5) with $m(x)w$, we have by (2.3)

$$\int_0^\pi m(x)(w(x))^2\,dx = \int_0^\pi\int_0^\pi G(x,\xi)m(\xi)w(\xi)m(x)w(x)\,d\xi\,dx \le \left(\int_0^\pi m(x)|w(x)|\,dx\right)^2.$$

Hence

$$\int_0^\pi m(x)|w(x)|\,dx \le \left(\int_0^\pi m(x)|w(x)|^2\,dx\right)^{\frac12}\left(\int_0^\pi m(x)\,dx\right)^{\frac12} \le \left(\int_0^\pi m(x)|w(x)|\,dx\right)\left(\int_0^\pi m(x)\,dx\right)^{\frac12}.$$

If $\int_0^\pi m(x)|w(x)|\,dx = 0$, then w is a nontrivial solution to (2.1) when $\lambda = 1$, and so $w = \beta\sin x$ for some $\beta\in\mathbb{R}\setminus\{0\}$; if $\int_0^\pi m(x)|w(x)|\,dx \ne 0$, then $\int_0^\pi m(x)\,dx \ge 1$. This proves the lemma.

THEOREM 2. *Let the problem* (1.1) *be given in which the nonlinearity g satisfies the conditions* (H1), (H2). *If there exist $r > 0$ and a, $b\in L^1(0,\pi)$ such that for a.e. $x\in(0,\pi)$*

$$g(x,u)\ge b(x)\quad \text{for } u\ge r, \qquad g(x,u)\le a(x)\quad \text{for } u\le -r, \tag{2.6}$$

for a.e. $x\in(0,z)$, then for any $h\in L^1(0,\pi)$, the problem (1.1) *is solvable provided that*

$$\int_0^\pi g_-(x)\sin x\,dx < \int_0^\pi h(x)\sin x\,dx < \int_0^\pi g_+(x)\sin x\,dx, \tag{2.7}$$

where $g_+(x) = \liminf_{u\to\infty} g(x,u)$, $g_-(x) = \limsup_{u\to-\infty} g(x,u)$ *for* $x\in(0,\pi)$.

PROOF. Let $\varepsilon > 0$ be chosen such that $\|\Gamma\|_{L^1} + \pi\varepsilon < 1$ and let $0<\gamma<\varepsilon$. We consider the boundary value problems

$$u'' + u + (1-t)\gamma u + \operatorname{tg}(x,u) = th(x) \quad \text{in } (0,\pi), \qquad u(0) = u(\pi) = 0 \tag{2.8}$$

for $0\le t\le 1$, which becomes the original problem (1.1) when $t = 1$. Since $0<\gamma<3$, (2.8) has only a trivial solution when $t=0$. We suppose for the moment that there exists $R>0$ such that $\|u\|_C < R$ for all possible solutions u to the problem (2.8) for some $0\le t\le 1$, and use this to finish proving the theorem. Let $K_{1+\gamma} : L^1(0,\pi)\to C[0,\pi]$ be the compact linear operator associated with the problem (2.4), where $\lambda = 1 + \gamma$ is not an eigenvalue of (2.1). We define $G : C[0, \pi] \to L(0, \pi)$ by

$$(Gu)(x) = h(x) + \gamma u(x) - g(x, u(x))$$

and $T = K_{1+\gamma} \circ G : C[0, \pi] \to C[0, \pi]$. Then G is continuous and maps bounded sets into bounded sets, and so T is a compact map. Clearly the problems (2.8) for $0 \le t \le 1$ are equivalent to the operator equations

$$u = tTu \tag{2.9}$$

for $0 \le t \le 1$, which by assumption have no solutions on the boundary of the ball $B_R(0) = \{u \in C[0, \pi] : \|u\|_C < R\}$. Thus the Leray-Schauder degree $\deg(I - tT, B_R(0), 0)$ is defined for $0 \le t \le 1$ and does not depend on t, where I denotes the identity operator. Hence $\deg(I - T, B_R(0), 0) = \deg(I, B_R(0), 0) = 1$, which implies that the operator equation $u = Tu$, or equivalently the original problem (1.1), has a solution in $B_R(0)$.

It remains to show that solutions to (2.8) for $0 < t \le 0$ have an a priori bound in $C[0, \pi]$. To this end, we first choose $r > 0$ such that

$$g(x, u)/u \le \Gamma(x) + \varepsilon \qquad \text{for } |u| \ge r \tag{2.10}$$

and the two inequalities in (2.6) hold. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $0 \le \theta(u) \le 1$ for $u \in \mathbb{R}$, $\theta(u) = 0$ for $|u| \le r$ and $\theta(u) = 1$ for $|u| \ge 2r$. We define

$$g_1(x, u) = \begin{cases} \min\{g(x, u) + |b(x)|,\ (\Gamma(x) + \varepsilon)u\}\theta(u) & \text{if } u \ge r \\ \max\{g(x, u) - |a(x)|,\ (\Gamma(x) + \varepsilon)u\}\theta(u) & \text{if } u \le -r \\ 0 & \text{if } |u| \le r \end{cases}$$

and $g_2(x, u) = g(x, u) - g_1(x, u)$. Then $g_1, g_2 : (0, \pi) \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Moreover, g_1 satisfies (H1),

$$0 \le g_1(x, u)/u \le \Gamma(x) + \varepsilon \tag{2.11}$$

for a.e. $x \in (0, \pi)$ and $u \in \mathbb{R}$, and g_2 is dominated by a function in $L^1(0, \pi)$, that is, $|g_2(x, u)| \le c(x)$ for some $c \in L^1(0, \pi)$ for a.e. $x \in (0, \pi)$ and $u \in \mathbb{R}$, where we define $g_1(x, u)/u = 0$ if $u = 0$. Now we argue by contradiction and suppose that there exist a sequence $\{u_n\}$ in $W^{2,1}(0, \pi) \cap H_0^1(0, \pi)$ and a corresponding sequence $\{t_n\}$ in $(0, 1)$ such that u_n is a solution to (2.8) when $t = t_n$ and $\|u_n\|_C \ge n$ for all n. Let $v_n = u_n/\|u_n\|_C$. Then $\|v_n\|_C = 1$ and

$$\begin{aligned} &v_n'' + (1 + m_n(x))v_n = h_n(x) \qquad \text{in } (0, \pi) \\ &v_n(0) = v_n(\pi) = 0, \end{aligned} \tag{2.12}$$

where

$$m_n(x) = (1-t_n)\gamma + t_n g_1(x, u_n(x))/u_n(x),$$
$$h_n(x) = (t_n h(x) - t_n g_2(x, u_n(x)))/\|u_n\|_C.$$

From the choice of γ and (2.11) we have

$$0 \le m_n(x) \le \Gamma(x) + \varepsilon \tag{2.13}$$

for a.e. $x \in (0,\pi)$. Clearly $\lim_{n\to\infty} h_n = 0$ in $L^1(0,\pi)$. By (2.13) the sequence $\{m_n\}$ is equi-integrable in $L^1(0,\pi)$ and so by the Dunford-Pettis theorem (see [2], p.76) it has a subsequence which converges weakly to a function m in $L^1(0,\pi)$. It follows from the Mazur theorem and the choice of $\varepsilon > 0$ that $0 \le m(x) \le \Gamma(x) + \varepsilon$ for a.e. $x \in (0,\pi)$ and so $\|m\|_{L^1} < 1$. By the compactness of the linear operator K_0 associated with the problem (2.4) when $\lambda = 0$, $\{v_n\}$ has a subsequence convergent in $C[0,\pi]$. From (2.12) it follows that $\{v_n''\}$ is dominated by a function in $L^1(0,\pi)$. Since each v_n' vanishes somewhere in $(0,\pi)$, the sequence $\{v_n'\}$ is equicontinuous and uniformly bounded on $[0,\pi]$ and so by the Ascoli theorem $\{v_n'\}$ has a subsequence convergent in $C[0,\pi]$. We may assume without loss of generality that $\{m_n\}$ converges weakly to m in $L^1(0,\pi)$, $t_n \to t_0$ and there exists $w \in C^1[0,\pi]$ such that $\{v_n\}$ converges to w in $C^1[0,\pi]$ and so also in $H_0^1(0,\pi)$. By (2.12) and the continuity of K_0 on $L^1(0,\pi)$ into $H_0^1(0,\pi)$, we have

$$w'' + (1 + m(x))w = 0 \qquad \text{in } (0,\pi), \quad w(0) = w(\pi) = 0. \tag{2.14}$$

Clearly $\|w\|_C = 1$. Moreover $t_0 \ne 0$, otherwise $m(x) = \gamma$ on $(0,\pi)$ and so (2.14) cannot have a non-trivial solution. Since $\|m\|_{L^1} < 1$, it follows from Lemma 1 that $w = \beta \sin x$ for some $\beta \in \mathbb{R} \setminus \{0\}$. We consider only the case $\beta > 0$; the alternative $\beta < 0$ can be treated similarly. Since $v_n = u_n/\|u_n\|_C \to w$ in $C^1[0,\pi]$, $u_n(x) \to \infty$ for $x \in (0,\pi)$. Moreover, by the elementary inequality

$$|v(x)/\sin x| \le (\pi/2) \max_{0\le\xi\le\pi} |v'(\xi)| \qquad \text{for } x \in [0,\pi]$$

held for all $v \in C^1[0,\pi]$ with $v(0) = v(\pi) = 0$, $u_n > 0$ on $(0,\pi)$ for n large enough. Taking the inner product in $L^2(0,\pi)$ of (2.8) when $u = u_n$ and $t = t_n$ with $\sin x$, we have

$$\begin{aligned} & t_n \int_0^\pi g(x, u_n(x)) \sin x \, dx \\ & \le (1-t_n)\gamma \int_0^\pi u_n(x) \sin x \, dx + t_n \int_0^\pi g(x, u_n(x)) \sin x \, dx \\ & = t_n \int_0^\pi h(x) \sin x \, dx. \end{aligned} \tag{2.15}$$

It follows from (H1) and the first inequality in (2.6) that $g(x, u_n(x))$ is bounded from below by a function in $L^1(0,\pi)$ independent of n for n large enough. Since

$t_0 \neq 0$, the Fatou lemma implies that

$$\int_0^\pi g_+(x) \sin x \, dx \leq \int_0^\pi h(x) \sin x \, dx$$

which contradicts the second inequality in (2.7). This completes the proof of the theorem.

3. A Variation

An interesting case in which the Landesman-Lazer condition (2.7) is not satisfied is when the equality holds in place of one of the inequalities. By modifying slightly the proof of Theorem 2 we obtain the following Theorem 3, which is an existence theorem for a solution to (1.1) without assuming a Landesman-Lazer condition. It can be applied to the problem (1.1) when the nonlinear term g is given, for example, by

$$g(x,u) = u^+ \log(\pi/x)/4$$

and $h \in L^1(0,\pi)$ satisfies $\int_0^\pi h(x) \sin x \, dx = 0$, where $u^+ = \max\{u, 0\}$. As mentioned in §1, Theorem 3 improves vastly the main result in [5] where it is assumed that $\|\Gamma\|_{L^1} \leq 1/15.87$.

THEOREM 3. *Let the problem* (1.1) *be given in which the nonlinearity g satisfies the conditions* (H1), (H2). *If*

$$g(x,u)u \geq 0 \qquad \text{for } u \in \mathbb{R}, \tag{3.1}$$

then the problem (1.1) *is solvable for any $h \in L^1(0,\pi)$ such that $\int_0^\pi h(x) \sin x \, dx = 0$.*

PROOF. In proving Theorem 2, the condition (2.7) is used only in the final part of the proof to produce contradictions. Thus we can proceed in exactly the same way as the proof of Theorem 2 up to the point where we choose the case $\beta > 0$ to consider and obtain (2.15). Then $u_n > 0$ on $(0,\pi)$ for n large enough. Since $0 < t_n < 1$, (2.15) implies

$$\int_0^\pi g(x, u_n(x)) \sin x \, dx < 0$$

which contradicts (3.1). This completes the proof of the theorem.

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