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PERIODIC SOLUTIONS OF AN ASYMPTOTICALLY LINEAR WAVE EQUATION

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Dedicated to the memory of Karol Borsuk

1. Introduction

This paper is concerned with the existence of solutions of the wave equation

$$\Box u := u_{tt} - u_{xx} = f(x,t,u), \qquad 0 < x < \pi, \ t \in \mathbf{R}, \tag{1}$$

satisfying the boundary and the periodicity conditions

$$u(0,t) = u(\pi,t) = 0, \qquad t \in \mathbf{R}, \tag{2}$$

$$u(x,t+2\pi) = u(x,t), \qquad 0 < x < \pi, \ t \in \mathbf{R}. \tag{3}$$

Suppose that f is continuous, 2π-periodic in t, strongly increasing (in the sense to be defined) and $f(x,t,\xi)/\xi \to b_0$ as $\xi \to 0, f(x,t,\xi)/\xi \to b$ as $|\xi| \to \infty$. Denote the spectrum of the operator $\Box$ subject to the conditions (2)–(3) by $\sigma(\Box)$. We will show that if $b_0, b \notin \sigma(\Box)$ and $\sigma(\Box)$ intersects the interval with the endpoints b_0 and b, then (1)–(3) has at least one non-trivial solution (in addition to the trivial one

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$u = 0$). The same conclusion remains valid under some additional hypotheses if $b_0, b \in \sigma(\Box)$.

Our result generalizes that of Amann and Zehnder [1, 2] where it was assumed that $f \in C^1$, the derivative f_ξ satisfies $0 < \alpha \le f_\xi \le \beta$ and $b \notin \sigma(\Box)$ (on the other hand, if $b_0 \in \sigma(\Box)$, our hypotheses at $\xi = 0$ are somewhat different and rather more restrictive). In [2] the assumptions that f_ξ is bounded and $b \notin \sigma(\Box)$ were used in order to reduce the problem to a finite dimensional one and to obtain estimates near infinity. Then it was shown via the Morse-Conley index theory that the reduced problem has a non-trivial solution.

In [4, 6] simpler proofs of the result of Amann and Zehnder were given: a finite dimensional reduction was used together with the classical Morse theory. Moreover, it was shown that under some additional hypotheses (1)–(3) possesses at least two non-trivial solutions.

Our approach here is different. We obtain solutions of (1)–(3) as critical points of a strongly indefinite functional in an L^2-space. To this functional we apply an infinite dimentional cohomology and a Morse theory developed in [17]. Since we make no finite dimensional reduction, we need not assume that $f \in C^1$ and $b \notin \sigma(\Box)$ (this second hypothesis was essential in obtaining the necessary estimates near infinity for the reduced functional).

Although we concentrate on the problems (1)–(3), we would like to mention that our method (with some obvious changes) also applies to the equation $\Box u = -f(x,t,u)$ with the boundary and the periodicity conditions (2)–(3).

In what follows the ball of radius r and center at the origin will be denoted by B_r, and its closure and boundary by $\overline{B}_r$ and ∂B_r, respectively. For a functional $\Phi \in C^1(E, \mathbf{R})$, E a real Hilbert space, we will use the customary notation

$$\Phi^a := \{u \in E : \Phi(u) \le a\} \quad \text{and} \quad K := \{u \in E : \nabla\Phi(u) = 0\}.$$

Recall that Φ is said to satisfy the *Palais-Smale condition* ((PS) in short) on a closed set A if each sequence $(u_n) \subset A$ such that $\Phi(u_n)$ is bounded and $\nabla\Phi(u_n) \to 0$ has a convergent subsequence. Recall also [5, 12] that the *critical groups* of an isolated critical point p of Φ are defined by

$$c_q(\Phi, p) := H_q(\Phi^a \cap U, \Phi^a \cap U - \{p\}), \tag{4}$$

where $a = \Phi(p)$, U is a closed neighbourhood of p and H_q is the q-th (singular) homology group with coefficients in a field $\mathcal{F}$ (c_q is independent of U by the excision property of homology). Since $c_q(\Phi, p) \approx 0$ for all q if Φ is strongly indefinite, we will also use a different notion of critical group which has been introduced in [17] and will be recalled in Section 2 of this paper.

2. An infinite dimensional cohomology and Morse theory

In this section we summarize some results from [17] and prove an additional result on cohomology of a saddle point.

Let E be a real Hilbert space with inner product $\langle,\rangle$ and suppose that $E = F \oplus \widetilde{F}$, where F and $\widetilde{F}$ are orthogonal to each other and F has an orthonormal basis $(e_n)_{n\in N}$. Denote

$$E_n := \{u = (x_1e_1 + \cdots + x_ne_n) + y : x_i \in \mathbf{R} \text{ for } 1 \le i \le n \text{ and } y \in \widetilde{F}\}$$

and

$$E_n^+ := \{u \in E_n : x_n \ge 0\}, \quad E_n^- := \{u \in E_n : x_n \le 0\}.$$

Observe that $E_n = E_n^+ \cup E_n^-$ and $E_{n-1} = E_n^+ \cap E_n^-$. For the pair (X, A) of closed subsets of E with $A \subset X$ denote $X_n := X \cap E_n$, $X_n^\pm := X \cap E_n^\pm$, and similarly for A. Then $(X_n; X_n^+, X_n^-)$ is a triad and $(A_n; A_n^+, A_n^-)$ its subtriad. Since $X_{n+1}^+ \cap X_{n+1}^- = X_n$ and $A_{n+1}^+ \cap A_{n+1}^- = A_n$, there exists an exact Mayer-Vietoris sequence of this pair of triads and a corresponding Mayer-Vietoris homomorphism

$$\triangle_n^q : H^{q+n}(X_n, A_n) \to H^{q+n+1}(X_{n+1}, A_{n+1}).$$

Here H^* is the Čech cohomology with coefficients in some field $\mathcal{F}$. Following an idea of Gęba and Granas [7], we define for each $q \in \mathbf{Z}$ a new cohomology group by

$$H_F^q(X, A) := \varinjlim_n \{H^{q+n}(X_n, A_n), \triangle_n^q\}.$$

A mapping $f : (X, A) \to (Y, B)$ is said to be *admissible* if $f_n := f|_{X_n}$ maps the triad $(X_n; X_n^+, X_n^-)$ to $(E_n; E_n^+, E_n^-)$ for almost all n. Then $\{f_n^*\}$ is a direct system of homomorphisms and

$$f^* := \varinjlim_n f_n^* : H_F^q(Y, B) \to H_F^q(X, A).$$

Admissible homotopies are defined in a similar way. There exists a coboundary homomorphism δ^* for the pair (X, A) and one shows [17, Theorem 2.2] that the cohomology theory H_F^* satisfies all the Eilenberg-Steenrod axioms except the dimension axiom which takes the following form: If $S := \{u \in F : \|u\| = 1\}$, then $H_F^{q-1}(S) \approx H^q(\{\text{point}\})$. Moreover, H_F^* has the strong excision property, i.e., $H_F^*(A \cup B, B) \approx H_F^*(A, A \cap B)$ whenever A and B are closed subsets of E.

Denote the orthogonal projectors of E onto F and E_n by P_F and P_n respectively. Suppose that $\Phi \in C^1(E, \mathbf{R})$ is a functional satisfying the following hypothesis:

(H) $\Phi(u) = \frac{1}{2}\langle Lu, u\rangle + \psi(u)$, where

(i) $L : E \to E$ is a linear, self-adjoint and bounded operator such that $LF \subset F$ and $LE_n \subset E_n \forall n \in \mathbf{N}$;

(ii) the gradient $\nabla\psi$ is bounded on bounded sets and the mapping $P_F\nabla\psi$ is compact.

A mapping $V : E - K \to E$ (where K is the set of critical points of Φ) is said to be a *pseudo-gradient vector field* for Φ if V is locally Lipschitz continuous and there exist constants $0 < \beta < \alpha$ such that

$$\|V(u)\| \leq \alpha\|\nabla\Phi(u)\| \quad \text{and} \quad \langle\nabla\Phi(u), V(u)\rangle \geq \beta\|\nabla\Phi(u)\|^2 \quad \forall u \in E - K. \tag{5}$$

Let A be a compact subset of K. A pair of sets (W, W^-) is said to be an *admissible pair* for A and Φ if

(i) W, W^- are closed in E, $W^- \subset W$, A is in the interior of W and there are no other critical points in W;
(ii) $\Phi|_W$ is bounded below;
(iii) there exist a neighbourhood N (in E) of the boundary ∂W of W and a pseudo-gradient vector field V for Φ on N such that $W - N$ is bounded and $V(u) = Lu + C(u)$, where C is bounded on bounded sets and $C(N) \subset E_{n_0}$ for some n_0;
(iv) W^- is the union of finitely many (possibly intersecting) closed sets each of which lies on a C^1-manifold of codimension 1 in E, V is transversal to each of these manifolds at points of W^-, the flow η of $-V$ can leave W only via W^-, and if $u \in W^-$, then $\eta(t, u) \notin W$ for any $t > 0$.

For an isolated critical point p of Φ we define the *critical groups* $c_F^q(\Phi, p)$ of Φ at p by

$$c_F^q(\Phi, p) := H_F^q(W, W^-), \quad q \in \mathbf{Z}. \tag{6}$$

It has been shown in [17, Section 3] that if Φ satisfies (H) and (PS), then each neighbourhood of p contains an admissible pair and the definition (6) is independent of the choice of such a pair.

Proposition 2.1. [17, Example 2.1(ii)] *Let X be a closed linear subspace of E such that $X_n := X \cap E_n$ is $(n + d)$-dimensional whenever n is sufficiently large. Then for each $r > 0$,*

$$H_F^q(\overline{B}_r \cap X, \partial B_r \cap X) \approx \begin{cases} \mathcal{F} & \text{if } q = d, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2. *A functional Φ is said to satisfy the local linking condition at 0 if there exist an orthogonal decomposition $E = X \oplus Y$ and a constant $\rho > 0$ such that*

$$\Phi(u) \geq c := \Phi(0) \quad \text{for each } u \in B_\rho \cap Y$$

and

$$\Phi(u) \leq c \quad \text{for each } u \in B_\rho \cap X.$$

This is a variant of a condition that has been introduced in [10].

The following result extends [11, Theorem 2.1] to the case of strongly indefinite functionals:

THEOREM 2.3. *Suppose that Φ satisfies* (H), (PS) *and the local linking condition at* 0 *(with X and Y as above). Suppose also that* 0 *is an isolated critical point of Φ. If $Y \subset E_n$ and $\dim X_n = n+d$ for all sufficiently large n, then $c_F^d(\Phi, 0) \not\approx 0$.*

PROOF. We may assume that 0 is the only critical point of Φ in B_ρ. Let $0 < \delta < \rho$. We first briefly recall the construction of an admissible pair (W, W^-) with $W \subset B_\delta$ which has been given in [17, Proposition 3.3]. Let $\varepsilon > 0$ be small enough and let $0 < \delta_2 < \delta_1 < \delta/2$, where δ_1 is chosen so that $|\Phi(u) - c| < \varepsilon$ for all $u \in \overline{B}_{\delta_1}$. There exists a pseudo-gradient vector field $\widetilde{V} = L + C : E - K \to E$ such that C is bounded on bounded sets and $P_F C$ is compact. Then for all n_0 large enough, $V := L + P_{n_0} C$ is a pseudo-gradient field on $N := \overline{B}_\delta - B_{\delta_1}$ (and has the form given in (iii) of the definition of an admissible pair). Moreover, according to [17, Lemma 3.2], the constants α and β (cf. (5)) for V do not depend on n_0 but only on the corresponding constants for $\widetilde{V}$. Let ω be a locally Lipschitz continuous function such that $\omega = 0$ on $\overline{B}_{\delta_2}$ and $\omega = 1$ on N. Consider the initial value problem

$$\frac{d\eta}{dt} = -\omega(\eta)V(\eta), \quad \eta(0, u) = u,$$

and note that $u \mapsto \eta(t, u)$ is an admissible map (whenever defined) because $V(u) = Lu + P_{n_0}C(u)$, and therefore $\eta(t, u) \in E_n^\pm$ if $u \in E_n^\pm$ and $n > n_0$. Set

$$W := \{\eta(t, u) : t \geq 0, u \in \overline{B}_{\delta_1}, \Phi(\eta(t, u)) \geq c - \varepsilon\}$$

and

$$W^- := W \cap \{u \in E : \Phi(u) = c - \varepsilon\}.$$

Then (W, W^-) is an admissible pair and $W \subset B_\delta$. Note that if one chooses a smaller δ_2 and a larger n_0, then W and W^- may change, but (W, W^-) will remain an admissible pair.

Let $\gamma \in (0, \delta_1/2)$. We may assume that $\delta_2 < \gamma < \delta_1/2$, $\omega = 1$ on $\overline{B}_\delta - B_\gamma$ and, choosing a larger n_0 if necessary, that V is a pseudo-gradient vector field on $\overline{B}_\delta - B_\gamma$ (with the same constants α and β as above). Denote

$$A := \{\eta(t, u) : t \geq 0, u \in \partial B_{\delta_1} \cap X, \Phi(\eta(t, u)) \geq c - \varepsilon\}.$$

Then $A \subset W$. Suppose $u \in \partial B_{\delta_1} \cap X$. Since $\Phi(u) \leq c$, one sees as in the proof of [17, Proposition 3.3] that if $\eta(t, u) \in \partial B_{\delta_1/2}$, then

$$\Phi(\eta(t, u)) \leq c - \varepsilon_0,$$

where $\varepsilon_0 > 0$ is independent of u and n_0. Choosing γ small enough, $\Phi(u) > c - \varepsilon_0$ $\forall u \in \overline{B}_\gamma$. Therefore $\eta(t,u)$ cannot enter $\overline{B}_\gamma$ whenever $u \in A$. Using (iv) of the definition of an admissible pair it follows that if $u \in A$, then there exists a unique $t(u)$, continuously depending on u, such that $\eta(t(u),u) \in W^-$. Hence the mapping

$$\alpha(s,u) := \begin{cases} \eta(st(u),u) & \text{if } u \in A, 0 \le s \le 1, \\ u & \text{if } u \in W^-, 0 \le s \le 1, \end{cases}$$

is an admissible strong deformation retraction of $A \cup W^-$ onto W^-. So it follows from the exact sequence of the triple $(W, A \cup W^-, W^-)$ that

$$H_F^*(W, W^-) \approx H_F^*(W, A \cup W^-). \tag{7}$$

For $y \in Y$, let

$$\delta_1(y) := \min\{\delta_1, d(y, A \cup W^-)\},$$

where $d(y, A \cup W^-)$ is the distance from y to the set $A \cup W^-$, and denote

$$D := \{x + y \in X \oplus Y : \|x\| < \delta_1(y)\}.$$

Since $(A \cup W^-) \cap Y = \emptyset$, D is an open set and $(A \cup W^-) \cap D = \emptyset$. Define

$$F_\delta := (\overline{B}_\delta \cap X) \oplus (\overline{B}_\delta \cap Y)$$

and let $i : (\overline{B}_{\delta_1} \cap X, \partial B_{\delta_1} \cap X) \to (W, A \cup W^-)$ and $j : (W, A \cup W^-) \to (F_\delta, F_\delta - D)$ be the inclusion mappings. Then we have

$$H_F^*(F_\delta, F_\delta - D) \xrightarrow{j^*} H_F^*(W, A \cup W^-) \xrightarrow{i^*} H_F^*(\overline{B}_{\delta_1} \cap X, \partial B_{\delta_1} \cap X), \tag{8}$$

where i^*, j^* are the induced homomorphisms. It is easy to see that the mapping

$$\beta(t, x + y) := \begin{cases} \dfrac{2t\delta_1 x}{\max\{\|x\|, \delta_1(y)\}} + (1-2t)x + y, & 0 \le t \le \frac{1}{2}, \\[2ex] \dfrac{\delta_1 x}{\max\{\|x\|, \delta_1(y)\}} + (2-2t)y, & \frac{1}{2} \le t \le 1, \end{cases}$$

is a deformation of $(F_\delta, F_\delta - D)$ onto $(\overline{B}_{\delta_1} \cap X, \partial B_{\delta_1} \cap X)$. It is admissible since $Y \subset E_{n_0}$ for some n_0 and therefore $\beta(t, x+y) \in E_n^\pm$ whenever $x + y \in E_n^\pm$ and $n > n_0$. Moreover, the restriction of β to $[0,1] \times (\overline{B}_{\delta_1} \cap X, \partial B_{\delta_1} \cap X)$ is a homotopy between $\beta(1,.) \circ (ji)$ and the identity on $(\overline{B}_{\delta_1} \cap X, \partial B_{\delta_1} \cap X)$. Similarly, β is a homotopy between $(ji) \circ \beta(1,.)$ and the identity on $(F_\delta, F_\delta - D)$. So the inclusion mapping ji is an admissible homotopy equivalence, and it follows that i^*j^* in (8) is an isomorphism. In particular, $H_F^q(W, A \cup W^-) \not\approx 0$ (and $H_F^q(W, W^-) \not\approx 0$ by (7)) whenever $H_F^q(\overline{B}_{\delta_1} \cap X, \partial B_{\delta_1} \cap X) \not\approx 0$. Now it remains to observe that $H_F^d(\overline{B}_{\delta_1} \cap X, \partial B_{\delta_1} \cap X) \approx \mathcal{F}$ according to Proposition 2.1. □

3. Variational formulation

Consider the wave equation

$$\text{(9)}\qquad \begin{cases} \Box u := u_{tt} - u_{xx} = f(x,t,u), & 0 < x < \pi, t \in \mathbf{R}, \\ u(0,t) = u(\pi,t) = 0, & t \in \mathbf{R}, \\ u(x,t+2\pi) = u(x,t), & 0 < x < \pi, t \in \mathbf{R}, \end{cases}$$

with f satisfying the following hypotheses:

(f_1) $f \in C([0,\pi] \times \mathbf{R} \times \mathbf{R})$ and $f(x,t+2\pi,\xi) = f(x,t,\xi)$ for all x,t,ξ;

(f_2) there exists an $\varepsilon > 0$ such that $(f(x,t,\xi) - f(x,t,\eta))(\xi - \eta) \geq \varepsilon(\xi - \eta)^2$ for all x, t, ξ, η (strong monotonicity);

(f_3) $f(x,t,\xi) = b\xi + g(x,t,\xi)$, where g is a bounded function;

(f_4) $f(x,t,\xi) = b_0\xi + g_0(x,t,\xi)$, where $g_0(x,t,\xi)/\xi \to 0$ uniformly in x and t as $\xi \to 0$.

Note that $f(x,t,0) = 0$ according to (f_4), so $u = 0$ is a solution of (9) (the trivial solution). Note also that $b_0, b > 0$.

Henceforth we will assume that f satisfies (f_1) – (f_4) although the results of this section remain true under weaker hypotheses.

Let $\Omega := (0,\pi) \times (0,2\pi)$ and let E be the space of functions

$$u(x,t) = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} \sin jx e^{ikt}, \qquad c_{j,-k} = \bar{c}_{jk},$$

with the norm given by

$$\|u\|^2 := \pi^2 \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} |c_{jk}|^2.$$

Then E is a subspace of $L^2(\Omega)$. In what follows we identify the operator $\Box$ with its self-adjoint extension in E. There exists an orthogonal decomposition

$$E = N(\Box) \oplus R(\Box),$$

where $\mathrm{N}(\Box)$ is the (generalized) nullspace and $\mathrm{R}(\Box)$ is the range of $\Box$. Define

$$Au := \Box u - bu$$

and let $E = F^+ \oplus F^0 \oplus F^-$ be the orthogonal decomposition into subspaces corresponding to the positive, zero and negative part of the spectrum of $-A$ (note the minus sign). Let $(e_m^0)_{m=1}^{m_0}$ be an orthonormal basis for F^0, $(e_m^-)_{m=1}^{\infty}$ an orthonormal system of eigenfunctions of $-A$ in F^- (with corresponding eigenvalues λ_m^-) and $(e_m^+)_{m=1}^{\infty}$, $(f_m)_{m=1}^{\infty}$ an orthonormal system of eigenfunctions of $-A$ in F^+

(with corresponding eigenvalues λ_m^+ and b respectively). Note that $f_m \in N(\Box)$ while $e_m^\pm, e_m^0 \in R(\Box)$. Denote the orthogonal projector of E onto F^0 by P and set

$$Ru := \sum_{m=1}^{\infty} \frac{1}{\sqrt{\lambda_m^+}} c_m^+ e_m^+ + \sum_{m=1}^{\infty} \frac{1}{\sqrt{b}} d_m f_m \tag{10}$$

and

$$Su := \sum_{m=1}^{\infty} \frac{1}{\sqrt{-\lambda_m^-}} c_m^- e_m^-, \tag{11}$$

where $c_m^\pm, c_m^0, d_m$ are the Fourier coefficients of u with respect to the basis $(e_m^\pm, e_m^0, f_m)$. Then R, S are linear, bounded and self-adjoint operators in E. Since all eigenvalues λ_m^- are isolated, of finite multiplicity and $\lambda_m^- \to -\infty$ as $m \to \infty$, S is compact (note that R is not).

A function u is said to be a *weak solution* of (9) if $u \in E$ and

$$\int_\Omega u \Box \varphi \, dx \, dt = \int_\Omega f(x,t,u) \varphi \, dx \, dt$$

for all smooth $\varphi \in E$. Let

$$G(x,t,\xi) := \int_0^\xi g(x,t,s) ds, \qquad \widetilde{\psi}(u) := \int_\Omega G(x,t,u) dx dt$$

and

$$\Phi(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \widetilde{\psi}((R+P+S)u), \tag{12}$$

where $u = u^+ + u^0 + u^- \in F^+ \oplus F^0 \oplus F^-$. The functional Φ has been introduced by Hofer in [9], where it was also shown that critical points of Φ correspond to weak solutions of (9). For the sake of completeness, we include the proof.

Proposition 3.1. *$\Phi \in C^1(E, \mathbf{R})$ and if $\nabla\Phi(u) = 0$, then $w = (R+P+S)u$ is a weak solution of* (9).

Proof. That $\widetilde{\psi}$, and therefore also Φ, is continuously differentiable is a standard fact which follows from the boundedness of g [14, Appendix B].

Suppose that u is a critical point of Φ. Then

$$\nabla\Phi(u) = u^+ - u^- + (R+P+S)\nabla\widetilde{\psi}((R+P+S)u) = 0.$$

Applying $-R+P+S$ to this equation and setting $w = (R+P+S)u$ gives

$$(I-P)w = (-R^2 + P + S^2)\nabla\widetilde{\psi}(w)$$

(I denotes the identity mapping). Since λ_m^+, b are the positive and λ_m^- the negative eigenvalues of $-A$, $-R^2+P+S^2 = (A+P)^{-1}$. Hence

$$Aw = \nabla\widetilde{\psi}(w),$$

or

$$\int_\Omega (w\Box\varphi - bw\varphi)\,dx\,dt = \int_\Omega g(x,t,w)\varphi\,dx\,dt$$

for all smooth $\varphi \in E$. So w is a weak solution of (9). $\square$

PROPOSITION 3.2. (i) *Φ satisfies the hypothesis* (H) *with $F = F^-$ (and a corresponding orthonormal basis $(e_m^-)_{m=1}^\infty$), $Lu = u^+ - u^-$ and $\psi(u) = \widetilde{\psi}((R+P+S)u)$.*

(ii) *Φ satisfies* (PS) *if either $b \notin \sigma(\Box)$ or $G(x,t,\xi) \to \infty$ uniformly in x and t as $|\xi| \to \infty$.*

PROOF. (i) Since

$$P_{F^-}\nabla\psi(u) = P_{F^-}(R+P+S)\nabla\widetilde{\psi}((R+P+S)u) = S\nabla\widetilde{\psi}((R+P+S)u),$$

the conclusion follows immediately from (12) and the fact that S is compact.

(ii) Suppose that $b \in \sigma(\Box)$, i.e., $F^0 \neq \{0\}$ (the other case is simpler). Let $\Phi(u_n)$ be bounded and

$$\nabla\Phi(u_n) = u_n^+ - u_n^- + \nabla\psi(u_n) \to 0. \tag{13}$$

Since g is bounded, so is $\nabla\psi(E)$, and therefore also the sequences (u_n^+), (u_n^-). Hence we may assume after passing to a subsequence that

$$u_n^+ \to \overline{u}^+ \quad\text{and}\quad u_n^- \to \overline{u}^- \quad \text{weakly in } E.$$

Furthermore, $\psi(u_n)$ is bounded (because $\Phi(u_n)$ is). By the mean value theorem,

$$\psi(u_n^0) - \psi(u_n) \le \sup_{u\in E}\|\nabla\psi(u)\|\,\|u_n^+ + u_n^-\|.$$

Hence $\psi(u_n^0)$ is bounded above. Since $(R+P+S)u_n^0 = u_n^0$,

$$\psi(u_n^0) = \int_\Omega G(x,t,u_n^0)dxdt.$$

Since F^0 is finite dimensional and all $u^0 \in F^0$ have the unique continuation property (i.e., if $\operatorname{meas}\{(x,t) \in \Omega : u^0(x,t) = 0\} > 0$, then $u^0 \equiv 0$), it is easy to see that $\psi(u_n^0) \to \infty$ whenever $\|u_n^0\| \to \infty$. Thus (u_n^0) is bounded and $u_n^0 \to \overline{u}^0$ after passing to a subsequence.

Denote the orthogonal projector of E onto $R(\Box)$ by Q. Since

$$QRu = \sum_{m=1}^\infty \frac{1}{\sqrt{\lambda_m^+}} c_m^+ e_m^+$$

(cf. (10)), λ_m^+ are isolated eigenvalues of finite multiplicity and $\lambda_m^+ \to \infty$ as $m \to \infty$, QR is compact. Therefore also $Q(R+P+S)$ is compact, and it follows from (13) that

$Qu_n \to Q\overline{u}$ strongly and $(I-Q)u_n \to (I-Q)\overline{u}$ weakly in E. Let $w := (R+P+S)u$. Then

$$(I-Q)w = (I-Q)(R+P+S)u = (I-Q)Ru = \sum_{m=1}^{\infty} \frac{1}{\sqrt{b}} d_m f_m = \frac{1}{\sqrt{b}}(I-Q)u$$

according to (10), (11). Hence $(I-Q)u_n \to (I-Q)\overline{u}$ strongly if and only if $(I-Q)w_n \to (I-Q)\overline{w}$ strongly.

We complete the proof by showing that $w_n \to \overline{w}$ strongly. As in the proof of Proposition 3.1, we see that $\nabla\Phi(u_n) \to 0$ implies

$$Aw_n = \nabla\widetilde{\psi}(w_n) + \zeta_n,$$

where $\zeta_n \to 0$. Keeping in mind that $Aw = \Box w - bw$, we obtain

$$\Box w_n = f(x,t,w_n) + \zeta_n.$$

Denoting the inner product in E by $\langle , \rangle$ and using (f_2) gives

$$\begin{aligned}\langle \Box w_n, w_n - \overline{w}\rangle &= \int_\Omega f(x,t,w_n)(w_n - \overline{w})\,dx\,dt + \langle \zeta_n, w_n - \overline{w}\rangle \\ &\geq \int_\Omega f(x,t,\overline{w})(w_n - \overline{w})\,dx\,dt + \varepsilon\|w_n - \overline{w}\|^2 + \langle \zeta_n, w_n - \overline{w}\rangle.\end{aligned}$$

Now $w_n \to \overline{w}$ weakly, $Qw_n \to Q\overline{w}$ strongly, $\Box w_n$ is bounded (because $f(x,t,w_n)+\zeta_n$ is) and $\langle \Box w_n, w_n - \overline{w}\rangle = \langle \Box w_n, Qw_n - Q\overline{w}\rangle$. Hence the left-hand side above tends to zero as $n \to \infty$. Since also the first and the third term on the right-hand side tend to zero, $w_n \to \overline{w}$ strongly. □

4. Local linking

LEMMA 4.1. (i) *Let H be a Hilbert space and let $\chi_s \in C^1(H,\mathbf{R})$ for $0 \leq s \leq 1$. Suppose that p is the only critical point of χ_s in a ball*

$$B := \{u \in H : \|u - p\| < r\}$$

(r independent of s) and all χ_s satisfy (PS) *on $\overline{B}$. If the functions χ_s are continuous with respect to s in the $C^1(B)$-topology (i.e., $\sup_{u\in B}\{|\chi_s(u) - \chi_{s_0}(u)| + |\nabla\chi_s(u) - \nabla\chi_{s_0}(u)|\} \to 0$ as $s \to s_0$), then the critical groups $c_q(\chi_s, p)$ (cf. (4)) are independent of s.*

(ii) *Suppose that p is an isolated critical point of $\chi \in C^1(H,\mathbf{R})$ and χ satisfies* (PS) *on $\overline{B}$, where B is as above. Then $c_0(\chi, p) \not\approx 0$ if and only if χ has a local minimum at p.*

Part (i) of this lemma for $\chi \in C^2$ is due to Gromoll and Meyer [8]. See also [5, Corollary 6.1] and [12, Theorem 8.8]. The argument in [12] relies on Lemmas 8.1

and 8.3 there. Since they remain valid for $\chi \in C^1$ (see e.g. [15, Theorems 3.1 and 4.5]), so does the conclusion in (i). In [17, Corollary 3.6] a corresponding result has been proven for functions of class C^1 and the critical groups c^q_F.

It is clear that $c_0(\chi, p) \approx \mathcal{F}$ if p is a local minimum of χ. In [5, Example 1.4] and [12, Theorem 8.6] it is shown that if p is not a local minimum of $\chi \in C^2$, then $c_0(\chi, p) \approx 0$. Again, if one uses [15, Theorem 4.5], the same argument as in [5, 12] shows that it suffices to have $\chi \in C^1$.

PROPOSITION 4.2. *Suppose that f satisfies the conditions* (f_1)–(f_4) *and assume that either $b_0 \notin \sigma(\Box)$ or there exists a $\delta > 0$ such that $g_0(x,t,\xi)\xi \geq 0$ (or $g_0(x,t,\xi)\xi \leq 0$) for all $|\xi| \leq \delta$. If 0 is an isolated critical point of Φ, then Φ satisfies the local linking condition at 0 (in the sense of Definition 2.2).*

PROOF. Suppose that $b_0 \in \sigma(\Box)$ and $g_0(x,t,\xi)\xi \geq 0$ for $|\xi| \leq \delta$ (the other cases are similar). Let $G_0(x,t,\xi) := \int_0^\xi g_0(x,t,s)ds$,

$$\widetilde{\psi}_0(u) := \int_\Omega G_0(x,t,u)\,dx\,dt, \qquad \psi_0(u) := \widetilde{\psi}_0((R+P+S)u) \tag{14}$$

and

$$Bu := (b_0 - b)(R+P+S)^2 u. \tag{15}$$

Since $g(x,t,\xi) = (b_0 - b)\xi + g_0(x,t,\xi)$,

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\langle Bu, u\rangle + \psi_0(u) \equiv \frac{1}{2}\langle (L+B)u, u\rangle + \psi_0(u).$$

Furthermore,

$$\begin{aligned} Lu + Bu = {} & \sum_{m=1}^{\infty}\left(1 + \frac{b_0 - b}{\lambda_m^+}\right)c_m^+ e_m^+ + \sum_{m=1}^{\infty} \frac{b_0}{b} d_m f_m \\ & + \sum_{m=1}^{m_0}(b_0 - b)c_m^0 e_m^0 + \sum_{m=1}^{\infty}\left(-1 + \frac{b_0 - b}{-\lambda_m^-}\right)c_m^- e_m^-. \end{aligned} \tag{16}$$

Let $E = X \oplus Y$, where X corresponds to the negative and Y to the non-negative part of the spectrum of $L + B$ (if $g_0(x,t,\xi)\xi \leq 0$ for $|\xi| \leq \delta$, X corresponds to the non-positive and Y to the positive part of the spectrum of $L+B$). Note that $X \subset R(\Box)$.

We claim that $\Phi(u) \leq 0$ for all $u \in B_\rho \cap X$ whenever ρ is small enough. Let $\chi_s : X \to \mathbf{R}$ be given by

$$\chi_s(u) := \frac{1}{2}\langle (L+B)u, u\rangle + s\psi_0(u), \quad 0 \leq s \leq 1.$$

Suppose that $u \in B_\rho \cap X$ and

$$\nabla\chi_s(u) \equiv (L+B)u + sP_X\nabla\psi_0(u) = 0 \tag{17}$$

(P_X is the orthogonal projector onto X). Using the argument of Proposition 3.1 and setting $w := (R+P+S)u$, we obtain

$$\Box w = b_0 w + sP_X g_0(x,t,w).$$

Since $\|u\| < \rho$ and $|g_0(x,t,w)| \le c_1|w|$, $\|b_0 w + sP_X g_0(x,t,w)\| \le c_2\rho$, where the constants c_1, c_2 are independent of w and s. Since $w \in \mathrm{R}(\Box)$, it follows from well-known regularity results (see e.g. [3]) that

$$\|w\|_{L^\infty} \le c_3\|b_0 w + sP_X g_0(x,t,w)\| \le c_4\rho. \tag{18}$$

Choosing ρ small enough and employing (14), (17), (18) and (f_4) gives

$$c_5\|u\|^2 \le -\langle (L+B)u,u\rangle = s\langle \nabla\psi_0(u),u\rangle = s\int_\Omega g_0(x,t,w)w\,dx\,dt \le \frac{1}{2}c_5\|u\|^2.$$

Therefore $u=0$. It follows that 0 is the only critical point of χ_s in $B_\rho \cap X$. Since $P_X = P_X Q$ and

$$P_X\nabla\psi_0(u) = P_X Q(R+P+S)\nabla\widetilde{\psi}_0((R+P+S)u),$$

we see that $P_X\nabla\psi_0$ is compact and χ_s satisfies (PS) on $\overline{B}_\rho \cap X$. Hence according to Lemma 4.1, the critical groups $c_q(-\chi_s,0)$ are independent of s. Since $-\chi_0(u) = -\frac{1}{2}\langle (L+B)u,u\rangle$ is positive definite, it attains its minimum at 0. Therefore $c_0(-\chi_1,0) = c_0(-\chi_0,0) \not\approx 0$ and $-\chi_1$ has a local minimum at 0. Since $\chi_1 = \Phi|_X$, $\Phi(u) \le 0$ for each $u \in B_\rho \cap X$, possibly after choosing a smaller ρ.

It remains to show that $\Phi(u) \ge 0$ for all $u \in B_\rho \cap Y$. Let $\theta_s : Y \to \mathbf{R}$ be given by the formula

$$\theta_s(u) := \frac{1}{2}\langle (L+B)u,u\rangle + \frac{1}{2}(1-s)\langle B_1u,u\rangle + s\psi_0(u), \quad 0 \le s \le 1,$$

where $B_1 u := (R+P+S)^2u$ (cf. (15)). Since $P_Y B_1 = B_1 P_Y$,

$$\nabla\theta_s(u) = (L+B)u + (1-s)B_1u + sP_Y\nabla\psi_0(u).$$

Let $u \in B_\rho \cap Y$ be a critical point of θ_s. Then

$$\Box w = b_0 w + (1-s)w + sP_Y g_0(x,t,w), \tag{19}$$

where $w = (R+P+S)u$. Writing $w = v+z$, $v \in \mathrm{N}(\Box)$, $z \in \mathrm{R}(\Box)$, we see as in (18) that

$$\|z\|_{L^\infty} \le c_6\rho, \tag{20}$$

c_6 independent of s. Assume for the moment that

$$\|v\|_{L^\infty} \le c_7\rho, \tag{21}$$

where c_7 is independent of s (this will be proved in Lemma 4.3 below). Since u is a critical point of θ_s,

$$\begin{aligned}0 &= \langle (L+B)u,u\rangle + (1-s)\langle B_1 u,u\rangle + s\langle \nabla\psi_0(u),u\rangle \\ &= \langle (L+B)u,u\rangle + (1-s)\int_\Omega w^2\,dx\,dt + s\int_\Omega g_0(x,t,w)w\,dx\,dt.\end{aligned}$$

If ρ is small enough, then $\|w\|_{L^\infty} \le (c_6+c_7)\rho \le \delta$. Since $\langle (L+B)u,u\rangle$ is positive semidefinite on Y, all terms on the right-hand side above are non-negative. Clearly, if $s<1$, then $w=0$, and therefore also $u=0$ (because $R+P+S$ is injective). So 0 is the only critical point of θ_s in $B_\rho \cap Y$ for $0 \le s < 1$. Let $s=1$. Then $(L+B)u=0$ and $g_0(x,t,w)=0$ a.e. Hence

$$\nabla\Phi(u) = (L+B)u + (R+P+S)\nabla\widetilde{\psi}_0(w) = (L+B)u = 0.$$

Since 0 is an isolated critical point of Φ, $u=0$. Moreover, it is seen as in the proof of Proposition 3.2 that each θ_s satisfies (PS) on $\overline{B}_\rho \cap Y$. So by Lemma 4.1, $c_0(\theta_1,0) = c_0(\theta_0,0) \not\approx 0$ because θ_0 attains its minimum at 0. Therefore $\theta_1 = \Phi|_Y$ has a local minimum at 0 and $\Phi(u) \ge 0$ for each $u \in B_\rho \cap Y$ provided ρ is sufficiently small. □

LEMMA 4.3. *Let $w = v+z$, where $v \in N(\Box)$ and $z \in R(\Box)$. If w satisfies* (19) *and z satisfies* (20), *then v satisfies* (21).

PROOF. We adapt the argument of Rabinowitz [13, Lemma 3.7]. Denote

$$h_s(x,t,\xi) := b_0\xi + (1-s)\xi + s g_0(x,t,\xi).$$

It follows from (f_2) that

$$(h_s(x,t,\xi) - h_s(x,t,\eta))(\xi-\eta) \ge \varepsilon(\xi-\eta)^2 \quad \forall \xi,\eta,s. \tag{22}$$

Since $N(\Box) \subset Y$, (19) implies

$$\int_\Omega h_s(x,t,w)\varphi\,dx\,dt = 0 \quad \forall\varphi \in N(\Box),$$

or equivalently,

$$\int_\Omega (h_s(x,t,v+z) - h_s(x,t,z))\varphi\,dx\,dt = -\int_\Omega h_s(x,t,z)\varphi\,dx\,dt \quad \forall\varphi\in N(\Box). \tag{23}$$

Each $v \in N(\Box)$ can be represented as

$$v(x,t) = p(t+x) - p(t-x) =: v^+(x,t) - v^-(x,t), \tag{24}$$

where $\int_0^{2\pi} p(\tau)d\tau = 0$. A simple computation shows that if $\widetilde{v}(x,t) = \widetilde{p}(t+x) - \widetilde{p}(t-x)$ is another element of $\mathrm{N}(\Box)$, then

$$\int_\Omega p(t+x)\widetilde{p}(t-x)\,dx\,dt = 0. \tag{25}$$

Let v in (23) have the representation (24), let $M \ge 0$ and

$$q(\tau) := \begin{cases} 0 & \text{for } |\tau| \le M, \\ \tau - M & \text{for } \tau > M, \\ \tau + M & \text{for } \tau < -M. \end{cases}$$

Then

$$\varphi(x,t) := q(v^+(x,t)) - q(v^-(x,t)) \in \mathrm{N}(\Box)$$

and $\varphi(x,t) \ge 0$ if $v(x,t) \ge 0$, $\varphi(x,t) \le 0$ if $v(x,t) \le 0$. Using this and (22)–(23), we obtain

$$\begin{aligned} \varepsilon \int_\Omega v(q(v^+) - q(v^-))\,dx\,dt &\le \int_\Omega (h_s(x,t,v+z) - h_s(x,t,z))(q(v^+) - q(v^-))\,dx\,dt \\ &\le \|h_s(x,t,z)\|_{L^\infty} \int_\Omega (|q(v^+)| + |q(v^-)|)\,dx\,dt. \end{aligned}$$

Since $\tau q(\tau) \ge M|q(\tau)|$ for all τ, it follows from (25) that

$$\begin{aligned} \int_\Omega v(q(v^+) - q(v^-))\,dx\,dt &= \int_\Omega (v^+ - v^-)(q(v^+) - q(v^-))\,dx\,dt \\ &= \int_\Omega (v^+ q(v^+) + v^- q(v^-))\,dx\,dt \ge M \int_\Omega (|q(v^+)| + |q(v^-)|)\,dx\,dt. \end{aligned}$$

Therefore

$$M\varepsilon \int_\Omega (|q(v^+)| + |q(v^-)|)\,dx\,dt \le \|h_s(x,t,z)\|_{L^\infty} \int_\Omega (|q(v^+)| + |q(v^-)|)\,dx\,dt. \tag{26}$$

If v is not essentially bounded, the integral above is non-zero and $M\varepsilon \le \|h_s(x,t,z)\|_{L^\infty}$ $\forall M \ge 0$, a contradiction. So $v \in L^\infty(\Omega)$. Let $M = \frac{1}{2}\|v^\pm\|_{L^\infty}$. Then either $M = 0$ and (21) is trivially satisfied or the integral in (26) is non-zero. In the second case we have

$$\|v\|_{L^\infty} \le 2\|v \pm\|_{L^\infty} = 4M \le \frac{4}{\varepsilon}\|h_s(x,t,z)\|_{L^\infty}$$

and (21) follows from (20) and the fact that $|h_s(x,t,\xi)| \le c_8|\xi|$, where c_8 is independent of s. □

Corollary 4.4. (i) *Suppose $b_0 < b$. Then $X = F^- \oplus F^0 \oplus S$, where $S \subset F^+$. If $b_0 \notin \sigma(\Box)$ or $g_0(x,t,\xi)\xi \ge 0$ $\forall |\xi| \le \delta$, then the dimension of S equals the number of eigenvalues of $\Box$ (counted with their multiplicity) in the interval (b_0, b). If $b_0 \in \sigma(\Box)$ and $g_0(x,t,\xi)\xi \le 0$ $\forall |\xi| \le \delta$, the interval should be changed to $[b_0, b)$.*

(ii) *Suppose $b_0 > b$. Then $X \subset F^-$ and X is spanned by all but finitely many eigenfunctions e_m^-. If $b_0 \notin \sigma(\Box)$ or $g_0(x,t,\xi)\xi \ge 0$ $\forall |\xi| \le \delta$, then the codimension* *of X in F^- equals the number of eigenvalues of $\Box$ (counted with their multiplicity) in the interval $(b, b_0]$. If $b_0 \in \sigma(\Box)$ and $g_0(x,t,\xi)\xi \le 0$ $\forall|\xi| \le \delta$, the interval should be changed to (b, b_0).*

PROOF. (i) Clearly, $\lambda \in \sigma(-A) \equiv \sigma(-\Box + b)$ if and only if $b - \lambda \in \sigma(\Box)$.

Recall that X corresponds to the negative part of the spectrum of $L + B$ (cf. (16)) if $b_0 \notin \sigma(\Box)$ or $g_0(x,t,\xi)\xi \ge 0$ $\forall|\xi| \le \delta$, and to the non-positive part if $g_0(x,t,\xi)\xi \le 0$ $\forall|\xi| \le \delta$. It follows from (16) that in the first case $X = F^- \oplus F^0 \oplus S$, where S is spanned by those e_m^+ for which $\lambda_m^+ < b - b_0$. The number of them is equal to the number of eigenvalues of $-A$ in the interval $(0, b - b_0)$, or equivalently, to the number of eigenvalues of $\Box$ in (b_0, b). To obtain the non-positive part of the spectrum, one adds those e_m^+ for which $\lambda_m^+ = b - b_0$, that is, the eigenfunctions of $\Box$ corresponding to the eigenvalue b_0.

(ii) The negative (non-positive) space of $L+B$ is spanned by those e_m^- for which $\lambda_m^- < b - b_0$ ($\lambda_m^- \le b - b_0$). So the remaining eigenfunctions e_m^- correspond to the eigenvalues $\lambda_m^- \in [b - b_0, 0)$ (respectively $\lambda_m^- \in (b - b_0, 0)$). Hence $b - \lambda_m^- \in (b, b_0]$ (respectively $b - \lambda_m^- \in (b, b_0)$). $\Box$

5. The main result

We are able now to formulate the main result of this paper.

THEOREM 5.1. *Suppose that f satisfies the conditions* (f_1)–(f_4) *and assume that either $b \notin \sigma(\Box)$ or $G(x,t,\xi) \to \infty$ uniformly in x and t as $|\xi| \to \infty$. Then the wave equation* (9) *has a non-trivial solution in each of the following cases:*

- (i) *$b_0 < b$, $b_0 \notin \sigma(\Box)$ and $(b_0, b] \cap \sigma(\Box) \ne \emptyset$;*
- (ii) *$b_0 < b$, $b_0 \in \sigma(\Box)$, $(b_0, b] \cap \sigma(\Box) \ne \emptyset$ and there is a $\delta > 0$ such that $g_0(x,t,\xi)\xi \ge 0$ $\forall|\xi| \le \delta$;*
- (iii) *$b_0 < b$, $b_0 \in \sigma(\Box)$ and there is a $\delta > 0$ such that $g_0(x,t,\xi)\xi \le 0$ $\forall|\xi| \le \delta$;*
- (iv) *$b_0 > b$, $b_0 \notin \sigma(\Box)$ and $(b, b_0) \cap \sigma(\Box) \ne \emptyset$;*
- (v) *$b_0 > b$, $b_0 \in \sigma(\Box)$, $(b, b_0) \cap \sigma(\Box) \ne \emptyset$ and there is a $\delta > 0$ such that $g_0(x,t,\xi)\xi \le 0$ $\forall|\xi| \le \delta$;*
- (vi) *$b_0 > b$, $b_0 \in \sigma(\Box)$ and there is a $\delta > 0$ such that $g_0(x,t,\xi)\xi \ge 0$ $\forall|\xi| \le \delta$.*

PROOF. By Proposition 3.1, critical points of Φ correspond to solutions of (9), and by Proposition 3.2, Φ satisfies (PS) and (H) with $F = F^-$, $Lu = u^+ - u^-$ and $\psi(u) = \widetilde{\psi}((R + P + S)u)$.

Suppose that 0 is the only critical point of Φ. Let

$$W := \{u \in E : \|u^-\| \le R\} \quad \text{and} \quad W^- := \{u \in E : \|u^-\| = R\}.$$

If R is large enough,

$$\begin{aligned}\langle \nabla\Phi(u), u^-\rangle &= -\|u^-\|^2 + \langle \nabla\psi(u), u^-\rangle \\ &\leq -\|u^-\|^2 + \|u^-\| \sup_{u\in E} \|\nabla\psi(u)\| < 0 \qquad \forall \|u^-\| \geq R.\end{aligned}$$

Moreover, if $u \in W$, then it follows from the mean value theorem that

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \psi(u^0) + (\psi(u) - \psi(u^0)) \\ &\geq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \psi(u^0) - (\|u^+\| + \|u^-\|) \sup_{u\in E} \|\nabla\psi(u)\| \\ &\geq \frac{1}{4}\|u^+\|^2 + \psi(u^0) - A,\end{aligned} \tag{27}$$

where A is a constant. Since $\psi(u^0) \to \infty$ as $\|u^0\| \to \infty$ (cf. the proof of Proposition 3.2), $\Phi|_W$ is bounded below and $\Phi(u) \to \infty$ as $\|u\| \to \infty$, $u \in W$.

Let $V = L + C$ be a pseudo-gradient vector field for Φ. Since $\nabla\psi$ has bounded range and $P_F\nabla\psi$ is compact (recall that $F = F^-$), C may be constructed in such a way that it has bounded range and $P_F C$ is compact (cf. [16, Proof of formula (3)] and [14, Appendix A]). Then

$$\langle Lu + P_n C(u), u^-\rangle = -\|u^-\|^2 + \langle P_n C(u), u^-\rangle < 0 \qquad \forall \|u^-\| \geq R \tag{28}$$

provided R is large enough (recall that P_n is the orthogonal projector onto E_n). It is easy to see [17, Lemma 3.2] that if N is a bounded set whose closure does not contain 0, then $L + P_n C$ is a pseudo-gradient vector field for Φ on N whenever n is sufficiently large. Let $a > \Phi(0) = 0$. Since $\Phi^a \cap W$ is a bounded set (cf. (27)), one sees using (28) that $(\Phi^a \cap W,\ \Phi^a \cap W^-)$ is an admissible pair for 0 and Φ and $L + P_n C$ (n large) is a corresponding pseudo-gradient vector field. Moreover,

$$H_F^*(\Phi^a \cap W, \Phi^a \cap W^-) \approx H_F^*(W, W^-). \tag{29}$$

This has been shown in the course of the proof of Theorem 7.2 in [17]. Therefore we only sketch the argument here. By excision, $H_F^*(\Phi^a \cap W, \Phi^a \cap W^-) \approx H_F^*((\Phi^a \cap W) \cup W^-, W^-)$. Let

$$D := \{u^+ + u^0 + u^- \in F^+ \oplus F^0 \oplus F^- : \|u^-\| \leq R, \|u^0\| \leq R_0, \|u^+\| \leq R_0\},$$

where R_0 is chosen so that $\Phi^a \cap W \subset D$. For $u \in W - D$, let $\gamma(t,u) := e^{-t}(u^+ + u^0) + e^t u^-$. There exists a unique $t(u) \geq 0$, continuously depending on u, such that $\gamma(t(u), u) \in \partial(W - D)$. Define

$$r_1(s,u) := \begin{cases} \gamma(st(u), u) & \text{if } u \in W - D, \\ u & \text{if } u \in D. \end{cases}$$

Then r_1 is an admissible strong deformation retraction of the pair (W, W^-) onto $(D \cup W^-, W^-)$. The flow of $-L - P_n C$ now provides an admissible strong deformation retraction r_2 of the pair $(D \cup W^-, W^-)$ onto $((\Phi^a \cap W) \cup W^-, W^-)$. It follows from (29) that

$$c_F^q(\Phi, 0) \approx H_F^q(W, W^-) \approx \begin{cases} \mathcal{F} & \text{if } q = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $H_F^*(W, W^-)$ was computed using Proposition 2.1 with $d = 0$ and the fact that the mapping $u \mapsto (1-s)u + su^-$ is an admissible strong deformation retraction of (W, W^-) onto $(W \cap F, W^- \cap F)$.

We complete the proof by showing that $c_F^q(\Phi, 0) \not\approx 0$ for some $q \neq 0$. According to Proposition 4.2, Φ satisfies the local linking condition at 0. If the hypothesis (i) or (ii) of the theorem is satisfied, it follows from Corollary 4.4 (i) that $Y \subset E_n$ and $\dim X_n = \dim(X \cap E_n) = n + d$ for all n, where $d := \dim(F^0 \oplus S)$ is the number of eigenvalues of $\square$ in the interval $(b_0, b]$. So $d > 0$ and Theorem 2.3 implies that $c_F^d(\Phi, 0) \not\approx 0$. If (iii) is satisfied, d is the number of eigenvalues in $[b_0, b]$. Since $b_0 \in \sigma(\square)$, $d > 0$.

Suppose now that the hypothesis (iv) or (v) is satisfied. Then by Corollary 4.4 (ii), $X \subset F^- \equiv F$, $Y \subset E_n$ and $\dim X_n = n + d$ for almost all n, where $-d := \operatorname{codim}_F X$ is the number of eigenvalues of $\square$ in (b, b_0). So $d < 0$ and $c_F^d(\Phi, 0) \not\approx 0$ according to Theorem 2.3. Finally, if (vi) is satisfied, $-d$ is the number of eigenvalues in $(b, b_0]$ and $d < 0$ because $b_0 \in \sigma(\square)$. □

COROLLARY 5.2. *If $b \notin \sigma(\square)$, then the hypothesis* (f_3) *in Theorem* 5.1 *may be replaced by*

(f_3') *$f(x, t, \xi) = b\xi + g(x, t, \xi)$, where $|g(x, t, \xi)| \leq \alpha|\xi| + \beta$ and $\alpha < d(b, \sigma(\square))$ (d denotes the distance).*

PROOF. We have

$$\alpha < d(b, \sigma(\square)) = d(0, \sigma(-A)) = \min\{|\lambda_m^\pm|, b\} =: \alpha_0.$$

Since $F^0 = \{0\}$, $R + P + S = R + S$, and it follows from (10)–(11) that

$$\begin{aligned} |\langle \nabla\psi(u), v\rangle| &= \left| \int_\Omega g(x, t, (R+S)u)(R+S)v \, dx \, dt \right| \\ &\leq \int_\Omega (\alpha|(R+S)u| + \beta)|(R+S)v| \, dx \, dt \leq \frac{\alpha}{\alpha_0}\|u\| \, \|v\| + D\|v\| \quad \forall v \in E, \end{aligned}$$

where D is a constant. If $\nabla\Phi(u) \equiv u^+ - u^- + \nabla\psi(u) = 0$, then

$$\|u\|^2 \equiv \|u^+\|^2 + \|u^-\|^2 = \langle \nabla\psi(u), u^- - u^+\rangle \leq \frac{\alpha}{\alpha_0}\|u\|^2 + D\|u\|.$$

So

$$\|u\| \leq \Big(1 - \frac{\alpha}{\alpha_0}\Big)^{-1} D. \tag{30}$$

Let

$$g_K(x,t,\xi) := \begin{cases} g(x,t,\xi) & \text{if } |\xi| \leq K, \\ g(x,t,K) & \text{if } \xi > K, \\ g(x,t,-K) & \text{if } \xi < -K, \end{cases}$$

and $f_K(x,t,\xi) := b\xi + g_K(x,t,\xi)$. Then f_K satisfies $(f_1) - (f_4)$ and (f_3'). Denote the corresponding functional by Φ_K and suppose $\nabla\Phi_K(u) = 0$. Set $(R+S)u = w = v + z$, where $v \in \mathrm{N}(\Box)$ and $z \in \mathrm{R}(\Box)$. Then

$$\Box w = bw + g_K(x,t,w) \equiv f_K(x,t,w).$$

As in (18) and (20) we see using (f_3') and (30) that

$$\|z\|_{L^\infty} \leq a_1\|bw + g_K(x,t,w)\| \leq a_2\|w\| + a_3 \leq a_4\|u\| + a_3 \leq a_5.$$

The same argument as in Lemma 4.3 (with h_s replaced by f_K) shows that

$$\|v\|_{L^\infty} \leq a_6.$$

The constants a_5 and a_6 are independent of K. Therefore choosing K large enough, $g_K(x,t,w) = g(x,t,w)$ and w is a (weak) solution of (9). □

Remarks. (i) The argument at the end of the proof of Corollary 5.2 (with g_K and f_K replaced by g and f) shows that each non-trivial weak solution w of (9) (with f satisfying (f_1)–(f_4)) is in $L^\infty(\Omega)$.

(ii) If $b_0 \notin \sigma(\Box)$, our argument may be simplified. Let

$$\Phi_s(u) := \frac{1}{2}\langle (L+B)u, u\rangle + s\psi_0(u), \quad 0 \leq s \leq 1.$$

Using the fact that $L + B$ is invertible, an argument similar to but simpler than the one in the proof of Proposition 4.2 shows that $u = 0$ is the only critical point of Φ_s in B_ρ if $\rho > 0$ is small enough. Since $\Phi_1 = \Phi$, it follows from [17, Corollary 3.6] that $c_F^*(\Phi, 0) = c_F^*(\Phi_0, 0)$. Since Φ_0 is a quadratic functional, it is easy to find an admissible pair for 0 and Φ_0 (cf. [17, Theorem 4.2]). Using Proposition 2.1 one shows that:

if $b_0 < b$, then $c_F^q(\Phi_0, 0) \not\approx 0$ if and only if $q = \dim(F^0 \oplus S)$,

if $b_0 > b$, then $c_F^q(\Phi_0, 0) \not\approx 0$ if and only if $q = -\operatorname{codim}_F X$,

where S and X are as in Corollary 4.4.

(iii) Suppose $b \notin \sigma(\Box)$ and the non-trivial critical point u_0 found in Theorem 5.1 is non-degenerate in the sense that $c_F^{q_0}(\Phi, u_0) \approx \mathcal{F}$ for some $q_0 \in \mathbf{Z}$ and $c_F^q(\Phi, u_0) \approx$

0 for $q \neq q_0$. Then Φ has another non-trivial critical point. Indeed, otherwise we obtain from the Morse inequalities [17, Theorem 5.1] that

$$t^{q_0} + t^{q_1} = 1 + (1+t)Q(t), \tag{31}$$

where

$$Q(t) = \sum_{q=-\infty}^{\infty} a_q t^q, \; a_q \geq 0$$

for all q and $a_q = 0$ for all $|q|$ large. Here t^{q_0} is the contribution of u_0 and t^{q_1} the contribution of $u = 0$ to the Morse polynomial. Now choosing $t = -1$ in (31) gives a contradiction.

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