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EXAMPLES RELATED TO ULAM'S FIXED POINT PROBLEM

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(*Submitted by A. Granas*)

Dedicated to the memory of Karol Borsuk

1. Introduction

In 1935, S. Ulam posed a problem which is listed as Problem 110 in the Scottish Book (see [13] or [15]): "Let M be a given manifold. Does there exist a numerical constant K such that every continuous mapping f of the manifold M into part of itself which satisfies the condition $|f^n(x) - x| < K$ for $n = 1, 2, \ldots,$ (where $f^n(x)$ denotes the n-th iteration of the image) possesses a fixed point: $f(x_0) = x_0$? The same under more general assumptions about M (general continuum?)."

Let X be a metric space and let ε be a positive number. A homeomorphism $h_\varepsilon : X \to X$ will be called an *Ulam ε-homeomorphism* if h_ε is fixed point free and, for every $x \in X$, the orbit $\{h_\varepsilon^n(x)\}_{n=1}^\infty$ is bounded by ε. We will say that X has the *property of Ulam* if X admits an Ulam ε-homeomorphism for every $\varepsilon > 0$.

A *continuum* is a nonempty compact connected metric space. A homeomorphism $h : X \to X$ is *periodic* if there exists an integer $k > 1$ such that for every $x \in X$, $h^k(x) = x$, and the smallest integer k with this property is called the *period* of h. A homeomorphism $h : X \to X$ is an *involution* if h is periodic with period $k = 2$.

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The most natural example of a continuum with the property of Ulam is the Cartesian product of infinitely many circles. Among the finite dimensional continua, the k-adic solenoid satisfies the property of Ulam: for every $\varepsilon > 0$, the k-adic solenoid admits an Ulam ε-homeomorphism of period k.

A continuum is *decomposable* if it is a union of two proper subcontinua. If $k > 1$, then the k-adic solenoid is not decomposable. A continuum is *nondegenerate* if it contains more than one point. A continuum is *hereditarily decomposable* if every nondegenerate subcontinuum is decomposable. Among examples described in this paper, there is a one-dimensional hereditarily decomposable continuum having the property of Ulam with h_ε being an involution for every $\varepsilon > 0$.

A *dynamical system* Φ on a metric space X is a continuous mapping $\Phi : X \times \mathbb{R} \to X$ (where $\mathbb{R}$ is the set of real numbers) such that for each $t \in \mathbb{R}$, $\Phi(X \times \{t\}) = X$, and if t_1 and t_2 belong to $\mathbb{R}$ and $p \in X$, then $\Phi(\Phi(p, t_1), t_2) = \Phi(p, t_1 + t_2)$, and $\Phi(p, 0) = p$. If X is a differentiable manifold and if V is a vector field on X such that $\lim_{t\to 0}(\Phi(p,t) - p)/t = V(p)$ for every point $p \in X$, then we say that the dynamical system Φ is *generated* by V. For every $p \in X$, the set $\{\Phi(p,t) : t \in \mathbb{R}\}$ is called the *trajectory* of p. If the trajectory of p consists of the single point p, then p is a *rest point* of the dynamical system. Other basic notions of the theory of dynamical systems can be found in [1].

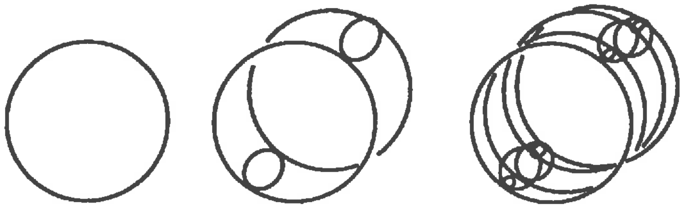
An Ulam ε-homeomorphism can arise from a rest point free dynamical system whose trajectories are uniformly bounded by ε. We will show that such a dynamical system can be constructed on an absolute neighborhood retract using Borsuk's example [4]; the space is the Cartesian product of the circle and the Hilbert cube. We also survey examples of rest point free dynamical systems on the Euclidean space $\mathbb{R}^3$ with uniformly bounded trajectories (see [11] and [12]) which settle Ulam's question for finite-dimensional manifolds.

The collection of examples of spaces that have the property of Ulam, presented in the following sections, includes two that are described here for the first time.

2. A one-dimensional, hereditarily decomposable continuum with fixed point free ε-involutions

Let S^1 denote the unit circle in the complex plane $\mathbb{C}$, *i.e.*, the set $\{z \in \mathbb{C} : |z| = 1\}$. In the product space $T = \prod_{j=1}^{\infty} S_j$ of a sequence of circles $S_j = S^1$ consider the subset X consisting of those points (z_j) whose at most one coordinate z_j is not equal to 1 or -1:

$$X = \bigcup_{k=1}^{\infty} \{(z_j) \in T : z_j = 1 \text{ or } z_j = -1 \text{ for } j \neq k\}.$$



X_1 X_2 X_3

FIGURE 1

Obviously, X is a closed subset of T. Let T_n be the product space $\prod_{j=1}^{n} S_j$ and regard T_n as a subset of T consisting of points (z_j) such that $z_j = 1$ for $j > n$. Let $X_n = T_n \cap X$. Figure 1 shows X_n for $n = 1, 2$, and 3.

Let $p_j : X_{j+1} \to X_j$ be the map obtained by deleting the $(j+1)$-st coordinate. The sequence of spaces $\{X_j\}$ with the bonding maps p_j form an inverse system, whose inverse limit is the same space X as we defined above. Since every X_j is a one-dimensional continuum, so is X. The hereditary decomposability of X is evident. Finally, for an arbitrary $\varepsilon > 0$, a fixed point free ε-involution on X can be obtained by changing the sign of the k-th coordinate of each point in X for a sufficiently large k and retaining the other coordinates without change.

REMARK 1. This continuum can be modified to produce a one-dimensional, locally connected (hence decomposable) continuum with the property of Ulam. The modification is accomplished by increasing the number of circular "bridges" connecting the two copies of X_j in X_{j+1}, so that the gaps between them tend to zero as $j \to \infty$. However, one cannot improve much more upon the local properties of this example. An addendum to Problem 110 in the Scottish Book indicates that the problem has an affirmative answer in the case of a locally contractible continuum of dimension smaller than or equal to two.

3. The Cartesian product of the circle and the Hilbert cube

In 1935 K. Borsuk [4] constructed an example of a cell-like, locally connected continuum in $\mathbb{R}^3$ which admits a fixed point free homeomorphism (a continuum is *cell-like* means it is the intersection of a nested sequence of cells). The Borsuk continuum is a solid cylinder with two tunnels carved out (see Figure 2). One tunnel starts at the top of the cylinder and spirals downwards to approach the circle bounding the bottom, and the other tunnel starts at the bottom and spirals upwards to approach the circle bounding the top. The width of each tunnel approaches zero as the tunnel gets close to the limiting circle. The spiraling tunnels are oriented so that for an observer looking from above a point falling down inside either of the two tunnels would rotate clockwise. Additionally, the entrance to each tunnel is a circular disk, concentric with the corresponding end of the cylinder, and the tunnel's descent (resp. ascent) is monotonic, so that each horizontal cross-section of the continuum is a circular disk with precisely two circular holes.

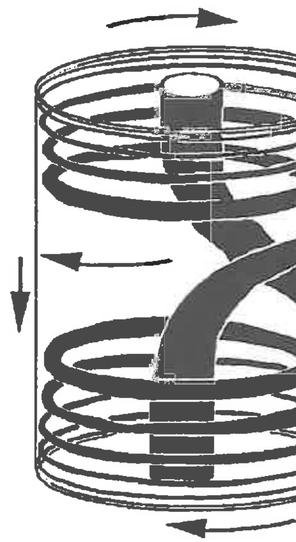


FIGURE 2

A fixed point free homeomorphism h on this continuum can be described as follows: We rotate clockwise and lower every horizontal cross-section except the top and bottom, which are rotated only. A horizontal cross-section lying strictly between the top and the bottom is transferred by h onto another, lower cross-section by a homeomorphism matching the rotation on the outer circle and accommodating the difference in sizes and positions of the two holes on the corresponding levels. No point remains fixed under this homeomorphism since the top and the bottom annuli rotate about the axis of the cylinder, and each point lying between the two extreme levels is moved to a level below its original position. The homeomorphism h can be modified slightly, without introducing any fixed points, so that it is a rotation not only on the top and bottom, but on the lateral surface of the cylinder as well. (For a reduction of Borsuk's example to a two-dimensional one see R. H. Bing [2] and [3].)

This ingenious example can be used as a building block to obtain a fixed point free, orientation preserving homeomorphism on $\mathbb{R}^3$ with bounded orbits. Such a homeomorphism is described by B. L. Brechner and R. D. Mauldin in [6]. A construction similar to that of [6] can also be used to produce rest point free dynamical systems on $\mathbb{R}^3$ with bounded trajectories. The "rotation" on Borsuk's continuum (by an arbitrary angle) can be extended to an embedding of the solid cylinder into $\mathbb{R}^3$ by pushing the inside of the first tunnel down into itself and pulling the inside of the first tunnel down. So extended "rotation" can be applied to a stack of cylinders (see Figure 3) such that the top of each of them is the bottom of another; the pulled-out part of one cylinder matching the pushed-in part of another. The trajectory of any point inside the stack is confined to two consecutive cylinders, and the points on the boundary of the stack rotate about its axis. This dynamical system on the stack extends to the remaining part of $\mathbb{R}^3$ as a rotation about the stack's axis, resulting in a rest point free dynamical system whose each trajectory is bounded.

The same idea can also be applied to a solid torus T (see Figure 4). By dividing T into k slices and applying the "extended rotation" to each of them, we get a fixed point free homeomorphism h_k of the torus onto itself with every orbit contained in the union of two neighboring slices. A map similar to h_1 can be derived from the dynamical system described by F. B. Fuller [8].

Let T be the Cartesian product of the circle S^1 and a disk D. If k is sufficiently large, we can choose the supremum of the diameters of the trajectories of h_k to be close to the diameter of D. Let X be the Cartesian product of S^1 and the Hilbert cube $Q = \prod_{i=1}^{\infty} D_i$, where D_i is a disk of diameter $1/i$. Let h_{kn} be the homeomorphism of X onto itself obtained by applying h_k to $S^1 \times D_n$ and the identity on the other factors of X. Observe that for every $\varepsilon > 0$, if n is greater than $2/\varepsilon$ and k is sufficiently large, then h_{kn} is an Ulam ε-homeomorphism.

The homeomorphism h_{kn} can easily be embedded in a rest point free dynamical system on $S^1 \times Q$.

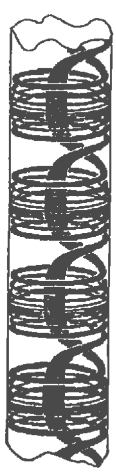


FIGURE 3

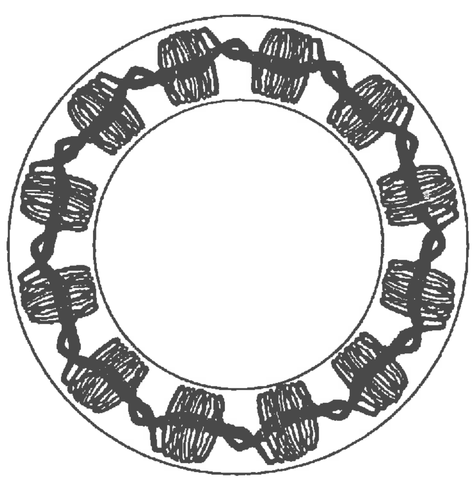


FIGURE 4

4. Rest point free dynamical systems with uniformly trajectories on three-dimensional manifolds

We take the unit cube to be the set

$$B = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}.$$

Suppose that Φ is a dynamical system generated by a vector field V defined on $\mathbb{R}^3$ and the following conditions are satisfied:

(*i*) There is a neighborhood N of the boundary of B such that if $p \in N$ or $p \notin B$, then $V(p) = (0, 0, 1)$.

(*ii*) If a trajectory of Φ passes through the points $(a, b, 0)$ and $(c, d, 1)$, then $(a, b) = (c, d)$.

(*iii*) Φ has no rest points.

(*iv*) There is a trajectory which intersects the boundary of B at exactly one point $(a, b, 0)$.

DEFINITION. The restriction to B of a vector field satisfying conditions (*i*) – (*iv*) above is called a *plug*.

Here are several types of plugs described in the literature:

1. A C^∞ plug constructed by F. W. Wilson [16]. The corresponding dynamical system Φ contains two circular trajectories.
2. A C^1 plug constructed by P. A. Schweitzer [14] for the purpose of solving the Seifert Conjecture (see Figure 5). The corresponding dynamical system Φ contains no circular trajectories.
3. A $C^{3-\varepsilon}$ modification of Schweitzer's plug due to J. Harrison [9].

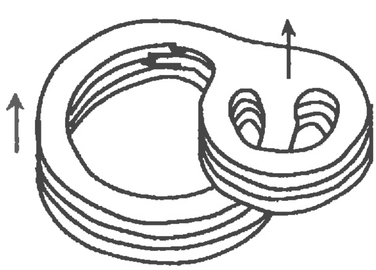


FIGURE 5

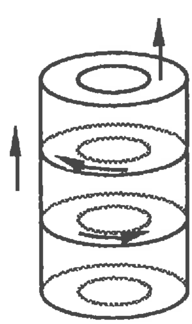


FIGURE 6

Both Wilson's and Schweitzer's plugs can be modified, without changing the differentiability properties, so that condition (*iv*) can be replaced by the following, stronger condition:

(*iv'*) There is an open set $U \subset \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$, such that every trajectory passing through a point $(a, b, 0)$ with $(a, b) \in U$ intersects the boundary of B in exactly one point.

Such a modification of Wilson's plug (see Figure 6) is used in [11] to solve Ulam's Problem 110 for manifolds, in that case for $\mathbb{R}^3$. (This modified plug contains two annuli of circular trajectories.) A similar example described in [12] is based on Schweitzer's plug, modified to satisfy condition (*iv'*). In both examples, for a given $\varepsilon > 0$, a dynamical system Ψ on $\mathbb{R}^3$ with trajectories uniformly bounded by ε is constructed. The homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $h(p) = \Psi(p, 1)$ has no fixed points, and its orbits are uniformly bounded by ε.

The above examples of dynamical systems on $\mathbb{R}^3$ are special cases of a more general construction:

Assume that V and Φ satisfy conditions (*i*), (*ii*), (*iii*), and (*iv'*). If Φ has circular trajectories, assume in addition that $\Phi(p, 1)$ has no fixed points.

Let U be the open set described in condition (*iv'*). Let δ be a positive number. Choose a family of open sets $\{U_i\}_{i=1}^n$, such that each U_i is a translation of U by the vector (a_i, b_i) and

$$\{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1\} \subset \bigcup_{i=1}^{n} U_i.$$

For a point $p = (x, y, z) \in \mathbb{R}^3$, let k and i be integers such that $1 \le i < n$ and $kn + (i - 1) \le z/\delta < kn + i$.

Define a vector field W on $\mathbb{R}^3$ by

$$W(p) = V((x/\delta - a_i - [x/\delta - a_i], y/\delta - b_i - [y/\delta - b_i], z/\delta - [z/\delta])),$$

where $[c]$ denotes the greatest integer less than or equal to c. Let Ψ be the dynamical system on $\mathbb{R}^3$ generated by W.

Condition (*iv'*) implies that if $(x, y) \in U$ and $p = (x, y, 0)$ $[p = (x, y, 1)]$, then there is a $t_0 \in \mathbb{R}$ such that $\Phi(p, t) \in B$ for $t > t_0$ $[t < t_0]$. Hence for every $p \in \mathbb{R}^3$, the length of the projection of the trajectory of p onto the z-axis is less than $(n+1)\delta$, and the projection of the trajectory of p onto the xy-plane is contained in a square with the side equal to 3δ. Therefore each trajectory is bounded by $(n + 1)^2\delta^2 + 9\delta^2 + 9\delta^{2^{1/2}} = \delta(n + 1)^2 + 18^{1/2}$.

To obtain a rest point free dynamical system Ψ on $\mathbb{R}^3$ whose trajectories are uniformly bounded by ε, take $\delta = \varepsilon(n + 1)^2 + 18^{-1/2}$. To show that $\mathbb{R}$ has the property of Ulam, *i.e.*, to obtain a fixed point free homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ whose orbits are uniformly bounded by ε, take $h(p) = \Psi(p, 1)$.

Note that if V is C^λ, $\lambda \leq \infty$, then W is C^λ, and if Φ has no circular trajectories, then Ψ has no circular trajectories.

QUESTION. *Does there exist a C^∞ [C^λ, $\lambda > 2$] dynamical system on $\mathbb{R}^3$ with uniformly bounded trajectories and no compact trajectories? In particular, can the ideas of Harrison* [9] *be used to construct such an example?*

REMARK 2. All closed 3-manifolds also have the property of Ulam. Every closed 3-manifold admits a rest point free dynamical system (see [16]), and the trajectories of any such dynamical system can be cut into small pieces by a large number of strategically placed small plugs, locally following the pattern described in the above construction.

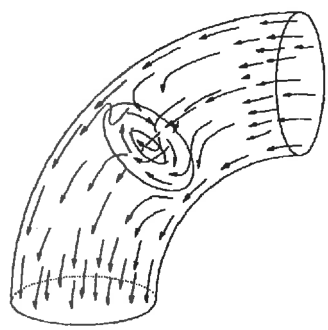


FIGURE 7

REMARK 3. As mentioned in §3., examples of C^∞ rest point free dynamical systems on $\mathbb{R}^3$ with bounded (but not uniformly bounded) trajectories can be derived from the results of Brechner and Mauldin, and of Fuller. An explicit example of such a dynamical system is described by G. S. Jones and J. A. Yorke in [10].

Jones and Yorke show that a differential equation may have all solutions bounded in $\mathbb{R}^3$ and have no critical points. They construct a dynamical system using an increasing sequence of solid tori $T_1, T_2, T_3, \ldots$, filling up $\mathbb{R}^3$ (see Figure 7). The boundaries of the tori rotate about the axes of revolution of the solid tori. Associated with each T_n, there is a positive number u_n, and a circle K_n with radius r_n and center p_n such that T_n is the closure of the u_n-neighborhood of K_n. The construction is carried out so that p_n is a point of K_{n+1}. If n is an even integer, then K_n lies in the xy-plane, and if n is odd, then K_n lies in the yz-plane. If p is a point of $T_{n+1} - T_n$ and d is the distance from p to K_{n+1}, then each point of the trajectory passing through p is at a distance d from K_{n+1}. If q is a point of $T_{n+1} - T_n$, and the distance between the trajectory passing through q and T_n is zero, then the α and ω limit sets of q are circles on the surface of T_n.

REMARK 4. A theorem of L. E. J. Brouwer [7] implies that there are no rest point free dynamical systems with bounded trajectories on $\mathbb{R}^2$. However, there is a fixed point free orientation reversing homeomorphism of $\mathbb{R}^2$ onto itself with all orbits of points bounded, see [5].

REFERENCES

[1] N. P. BHATIA AND G. P. SZEGÖ, *Stability theory of dynamical systems*, Springer-Verlag, 1970.

[2] R. H. BING, *Challenging conjectures*, Amer. Math. Monthly, 50th Anniversary Issue **74(II)** (1967), 56–64.

[3] ———, *The elusive fixed point property*, Amer. Math. Monthly **76** (1969), 119–132.

[4] K. BORSUK, *Sur un continu acyclique qui se laisse transformer topologiquement en lui même sans points invariants*, Fund. Math. **24** (1935), 51–58.

[5] S. BOYLES, *A counterexample to the bounded orbit conjecture*, Trans. Amer. Math. Soc. **266** (1981), 415–422.

[6] B. L. BRECHNER AND R. D. MAULDIN, *Homeomorphisms of the plane*, Pacific J. Math. **59** (1975), 375–381.

[7] L. E. J. BROUWER, *Beweis des ebenen Translationsatzes*, Math. Ann. **72** (1912), 39–54.

[8] F. B. FULLER, *Note on trajectories in a solid torus*, Ann. of Math. **56** (1952), 438–439.

[9] J. HARRISON, *A C^2 counterexample to the Seifert conjecture*, Topology **27** (1988), 249–278.

[10] G. S. JONES AND J. A. YORKE, *The existence and nonexistence of critical points in bounded flows*, J. Differential Equations **6** (1969), 238–246.

[11] K. M. KUPERBERG AND C. S. REED, *A rest point free dynamical system on R^3 with uniformly bounded trajectories*, Fund. Math. **114** (1981), 229–234.

[12] ———, *A dynamical system on R^3 with uniformly bounded trajectories and no compact trajectories*, Proc. Amer. Math. Soc. **106** (1989), 1095–1097.

[13] R. D. MAULDIN (ed.), *The Scottish Book: Mathematics from the Scottish Café*, Birkhäuser, 1981.

[14] P. A. SCHWEITZER, *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, Ann. of Math. **100** (1974), 386–400.

[15] S. M. ULAM, *The Scottish Book*, vol. LA-6832, Los Alamos Scientific Monograph.

[16] F. W. WILSON, *On the minimal sets of non-singular vector fields*, Ann. of Math. **84** (1966), 529–536.

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