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SUBHARMONIC SOLUTIONS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Karol Borsuk

1. Introduction and notations

We consider the system of second order differential equations

$$\ddot{x} + \nabla G(t, x(t)) = 0, \tag{1}$$

where $G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, periodic with minimal period $T > 0$ in its first variable and is such that its first and second derivatives with respect to its second variable $D_x G(t, x)$ and $D_x^2 G(t, x)$ are continuous; we shall write $\nabla G(t, x)$ for $D_x G(t, x)$. When $N = 1$ we write equation (1) in the form

$$\ddot{x}(t) + g(t, x(t)) = 0, \tag{2}$$

and, accordingly, we define

$$G(t, x) = \int_0^x g(t, s)\, ds.$$

Our purpose is to study the problem of the existence of kT-periodic solutions of (1) or (2) ($k \geq 1$ is an integer) which are not T-periodic. These solutions will be found by applying Morse theory to the associated functional

$$\varphi_k(x) = \int_0^{kT} \left(\frac{|\dot{x}(t)|^2}{2} - G(t, x(t)) \right) dt$$

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for $x \in H_k \equiv H^1_{kT}(\mathbb{R};\mathbb{R}^N)$, the Sobolev space consisting of the kT-periodic absolutely continuous functions $\alpha : \mathbb{R} \to \mathbb{R}^N$, whose first derivative is in $L^2([0,kT];\mathbb{R}^N)$, equipped with the usual inner product

$$\int_0^{kT} [(x(t),y(t)) + (\dot{x}(t),\dot{y}(t))]\,dt.$$

Here $(\cdot\,,\,\cdot)$ stands for the Euclidean inner product in $\mathbb{R}^N$ and $\|\cdot\| = (\cdot\,,\,\cdot)^{1/2}$. Identifying $\mathbb{R}^N$ with the space of constant functions we may write $H_k = \mathbb{R}^N \oplus \tilde{H}_k$ (orthogonal decomposition) and, for each $x \in H_k$,

$$x(t) = \bar{x} + \tilde{x}(t),$$

where

$$\bar{x} = \frac{1}{kT}\int_0^{kT} x(t)\,dt,$$

so that

$$\int_0^{kT} \tilde{x}(t)\,dt = 0.$$

We also consider the Banach space $C([0,T];\mathbb{R}^N)$ of continuous functions $x : [0,T] \to \mathbb{R}^N$ equipped with the norm $\|x\|_\infty = \sup_{0\le t\le T}|x(t)|$. We shall denote by $\|\cdot\|_2$ the usual L^2-norm.

It is well known that under our regularity assumptions the set of kT-periodic solutions of (1) coincide with the set of critical points of φ_k. Moreover, φ_k is a C^2 functional and $D^2\varphi_k(x)$ is a Fredholm operator, for each $x \in H_k$.

It is clear that a kT-periodic solution of (1), even if it is not T-periodic, needs not have minimal period kT. However, if for example k is a prime number and the property

(H_0) if $z(t)$ is a periodic function with minimal period qT, q rational, and $\nabla G(t,z(t))$ is a periodic function with minimal period qT, then q is necessarily an integer

holds, then any kT-periodic solution of (1) which is not T-periodic must have minimal period kT (see [11]); these are called subharmonic solutions of (1). For example, if $G(t,x) = a(t)G(x)$ or $G(t,x) = G(x) + (h(t),x)$, where $a(t) > 0$ and $h(t)$ have minimal period T, then (H_0) holds. Our main results (Theorem 1 and 2) state that under certain conditions upon the function $G(t,x)$ there exist kT-periodic solutions which are not T-periodic, for every k sufficiently large; under the additional assumption (H_0) this provides subharmonics for (1) with minimal period kT, for every k prime and large.

The typical case we consider is the convex subquadratic case (see theorems 3 and 4); this was studied in [11], [17] by the use of a $\mathbb{Z}_p$-index theory (in [17] the superquadratic case was also considered). In [17] it is assumed that (H_0) holds,

$G(t,\cdot)$ is convex for every $t \in \mathbb{R}$, $G \geq 0$, $G(t,0) = 0$ and there exist positive constants $a_1, a_2, a_3, a_4, \alpha, \mu$ with $1 < \alpha \leq \mu < 2$, such that

$$a_1|x|^{\alpha} - a_2 \leq G(t,x) \leq a_3|x|^{\mu} + a_4$$

for every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. However the examples given in section 3 show that the main ideas contained in theorems 1 and 2 may apply to other situations where neither convexity nor subquadratic growth hold. We also note that the periodic case (i.e., $G(t, x+\tau) = G(t,x)$ for some $\tau > 0$) was treated in [6] using some ideas developed in this paper.

The paper is organized as follows: in section 2 we recall some definitions for the estimate of the Morse index of critical points at critical levels of inf-sup and then prove our main abstract result. In section 3 we apply the ideas of section 2 to equation (1) and (2) in several different situations.

2. A general result

We start by recalling the following definitions: let X be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\varphi : X \to \mathbb{R}$ be a C^2 function. We let $D^2\varphi(x)$ denote the unique bounded self-adjoint operator in X such that $\langle D^2\varphi(x)y, z\rangle = \varphi''(x)(y)(z)$ for every $x, y, z \in X$. Let x_0 be a critical point of φ; we define the Morse index [augmented Morse index] $m_{\varphi}(x_0)$ [$m^*_{\varphi}(x_0)$] of x_0 as the supremum of the dimensions of the vector subspaces of X over which $D^2\varphi(x_0)$ is negative definite (semi-negitive definite). We also define the nullity $\nu_{\varphi}(x_0) = m^*_{\varphi}(x_0) - m_{\varphi}(x_0)$; x_0 is called nondegenerate if $\nu_{\varphi}(x_0) = 0$.

When applied to the functional φ_k, $k \geq 1$, defined in section 1, we simply write $m_k(x_0), m^*_k(x_0), \nu_k(x_0)$; in this case we can use the following alternative (equivalent) approach: for every $\sigma \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and every kT-periodic solution $x(t)$ of (1) define $J(kT,\sigma,x)$ [$J^*(kT,\sigma,x)$] as the number of negative [non positive] real numbers λ, counted with their multiplicity, for which there exists a nontrivial solution of the problem

$$\begin{aligned} &v''(t) + (D^2_x G(t,x(t)) + \lambda)v(t) = 0 \\ &v(t+kT) = \sigma v(t). \end{aligned}$$

Then $m_k(x) = J(kT,1,x)$ and $m^*_k(x) = J^*(kT,1,x)$; notice that $\nu_k \leq 2N$. Moreover, the function $J(kT,\cdot,\cdot) : S^1 \times H_k \to \mathbb{N}$ is lower semi-continuous (see[1]).

LEMMA 1. *Let $x(t)$ be a T-periodic (hence kT-periodic) solution of (1) such that $m_1(x) \geq 1$. Then*

$$\lim_{k\to\infty} m_k(x) = +\infty.$$

Proof. Consider the eigenvalue problem

$$v''(t) + (D_x^2 G(t, x(t)) + \lambda)v(t) = 0 \tag{3}$$

$$v(t + kT) = v(t). \tag{4}$$

By a result of Bott [2] there exists a non trivial solution of (3), (4) if and only if there exists a nontrivial solution $v(t)$ of (3) verifying

$$v(t + T) = \sigma v(t) \tag{5}$$

for some $\sigma \in S^1$, $\sigma^k = 1$ (see the proof of Proposition 2.1 (iv) in [1]). Besides, one has

$$m_k(x) = \sum_{\sigma^k=1} J(T, \sigma, x). \tag{6}$$

Assume $m_1(x) \geq 1$. Then also $J(T, \sigma, x) \geq 1$ for some $\sigma \in S^1$ and $|\sigma - 1| \leq \epsilon$ (for small $\epsilon > o$). Now, given $M \in \mathbb{N}$, choose $k_0 \geq M$ such that $|e^{2\pi ij/k_0} - 1| \leq \epsilon$ for every $j \in \{1, \ldots, M\}$. Then if $k \geq k_0$ we have $J(T, e^{2\pi ij/k}, x) \geq 1$, $j \in \{1, \ldots, M\}$. From (6) we get $m_k(x) \geq M$ and this proves the lemma. □

Remark 1. Let C be a compact subset of H_1 consisting of critical points of φ_1 such that $m_1(x) \geq 1$ for every $x \in C$. Then also $J(T, \sigma, x) \geq 1$ for every $x \in C$, $|\sigma - 1| \leq \epsilon$, $\sigma \in S^1$, if ϵ is small enough. The preceding argument then shows that $m_k(x) \to +\infty$ as $k \to \infty$, uniformly in $x \in C$.

Lemma 2. *Assume $N = 1$ and let $x(t)$ be a T-periodic solution of (2) such that $m_1(x) = 0$. Then*

$$m_k(x) = 0, \qquad \textit{for every } k \geq 1$$

and, either $\nu_k(x) = 0$ for every $k \geq 1$, or $\nu_k(x) = 1$ for every $k \geq 1$.

Proof. Denote by $\lambda_k(x)$ the first eigenvalue of (3), (4). It is clear that $\lambda_k(x) \leq \lambda_1(x)$ for every $k \geq 1$; but from the theory of Hill's equation (see [8]) one knows that situations (3), (5) belong to $]0, +\infty[\setminus\{1\}$ if $\lambda < \lambda_1(x)$). Hence we deduce that

$$\lambda_1(x) = \lambda_k(x)$$

for every $k \geq 1$. Now, from the very definition of Morse index we have $m_1(x) = 0$ if and only if $\lambda_1(x) \geq 0$ and so we have $m_k(x) = 0$ for every $k \geq 1$. If $\lambda_1(x) = 0$ then $\nu_k(x) = 1$ for every $k \geq 1$, since 0 is the first eigenvalue, which is simple. If otherwise $\lambda_1(x) > 0$, then $\nu_k(x) = 0$ for every $k \geq 1$. □

Next we recall two results which provide estimates for the Morse index of some class of critical points. Given a Hilbert space X and a C^2 function $\varphi : X \to \mathbb{R}$ we shall say that φ verifies the Palais-Smale condition (in short (PS) condition) over X if any sequence (x_n) in X such that $(\varphi(x_n))$ is bounded and $\nabla\varphi(x_n) \to 0$ has a convergent subsequence in X. For each $R > 0$ and $x \in X$ we denote by

$B_R(x)$ the open ball centered at x with radius R and by $S_R(x)$ its boundary. Also, we assume that $D^2\varphi(x)$ is a Fredholm operator for every critical point x of φ.

LEMMA 3 (MOUNTAIN PASS THEOREM). *Let φ be as above, assume that φ satisfies the (PS) condition over X and has only isolated critical points. Suppose that there exist $R > 0$ and $x_0, x_1 \in X$ such that $\|x_0 - x_1\| > R$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{S_R(x_0)} \varphi. \tag{7}$$

Then there exists a critical point x of φ such that $x \neq x_0$ and

$$m_\varphi(x) \leq 1 \leq m^*_\varphi(x).$$

(For a proof see [9]).

REMARK 2. Let us recall that condition (7) holds if x is an isolated local minimum of φ, provided φ satisfies the (PS) condition and $\varphi(u_n) \to -\infty$ for some unbounded sequence (u_n) in X (see [3, Theorem 5.10]).

LEMMA 4 (SADDLE POINT THEOREM). *Let φ be as above, assume that φ satisfies the (PS) condition over X and has only isolated critical points. Let $X = X_1 \oplus X_2$, X_1 and X_2 being closed subspaces of X with $\dim X_1 = d$, $1 \leq d < \infty$ and assume that for some $R > 0$ one has*

$$\sup_{S_R(0)\cap X_1} \varphi < \inf_{X_2} \varphi. \tag{8}$$

Then there exists a critical point x of φ such that

$$m_\varphi(x) \leq d \leq m^*_\varphi(x).$$

(See [7] or [9].)

From this we can deduce the following

LEMMA 5. *Assume $N = 1$ and for some $k > 1$ the functional φ_k associated to equation (2) satisfies the (PS) condition over H_k and the geometric assumption (8) of Lemma 4 [resp: (7) of Lemma 3]. Let d be as in Lemma 4 [resp: $d \equiv 1$ if (7) holds]. Moreover, assume that the (non empty) set Z of critical points of φ is compact in H_1 and that, for every $x \in Z$, either*

$$m_k(x) \geq d + 1 \tag{9}$$

or

$$m_k(x) = 0 = m^*_k(x). \tag{10}$$

Then equation (2) has a kT-periodic solution which is not T-periodic.

PROOF. Let Z_0 and Z_1 denote the subsets of Z whose points verify (10) and (9) respectively. From our hypothesis, Z_0 is finite and Z_1 is compact. Assume by contradiction that (2) has only T-periodic solutions. A compactness and continuity argument shows that we can fix positive constants α, β such that

$$x \in Z_0, |x - z| \leq \alpha \Rightarrow (D^2\varphi_k(z)h, h) \geq \beta |h|^2,$$

for every $h \in H_k$, and

$$x \in Z_1, |x - z| \leq \alpha \Rightarrow (D^2\varphi_k(z)h, h) \leq -\beta |h|^2,$$

for every $h \in E_x$, where E_x is some finite dimensional subspace of H_k with $\dim E_x \geq d + 1$. Here $(\,\cdot\,,\,\cdot\,)$ stands for the inner product in H_k.

Consider the situation (7), take x_0, x_1, R as in Lemma 3 and choose

$$0 < \epsilon < \min\{\beta/2,\ (\max\{\varphi_k(x_0), \varphi_k(x_1)\} \ - \inf_{S_R(x_0)} \varphi_k)/3\}.$$

According to a perturbation theorem of [10], we can choose a C^2 functional ψ such that

(i) ψ satisfies the (PS) condition over H_k;

(ii) ψ has only non degenerate critical points (in particular, they are isolated);

(iii) $\psi(z) = \varphi(z)$ whenever $\mathrm{dist}(z, Z) \geq \alpha/2$;

(iv) $\sup_{z \in H_k}\{|\varphi_k(z) - \psi(z)| + \|\nabla\varphi_k(z) - \nabla\psi(z)\|_{H_k} + \|D^2\varphi_k(z) - D^2\psi(z)\|_{\mathcal{L}(H_k)}\} < \epsilon.$

It follows from our choice of ϵ that

$$\max\{\psi(x_0), \psi(x_1)\} < \inf_{S_R(x_0)} \psi,$$

and from Lemma 3 we can take a critical point z of ψ with Morse index one. Now take $x \in Z$ such that $\|z - x\| = \mathrm{dist}(z, Z) \leq \alpha$. Then, either

$$(D^2\psi(z)h, h) \geq (\beta/2)|h|^2,$$

for every $h \in H_k$ (if $x \in Z_0$), or

$$(D^2\psi(z)h, h) \leq -(\beta/2)|h|^2,$$

for every $h \in E_x$ (if $x \in Z_1$). Since $\dim E_x \geq d + 1 \geq 2$ we get a contradiction in both cases.

Finally, if situation (8) holds, we proceed as before by choosing

$$0 < \epsilon < \min\{\beta/2,\ (\inf_{X_2} \varphi_k)/3 \ - \sup_{S_R(0) \cap X_1} \varphi_k\}$$

where R, X_1, X_2 are as in Lemma 4. □

Now we can state our main general results.

THEOREM 1. *Let $N = 1$, consider the functional φ_k $(k \geq 1)$ associated to equation (2) and assume that φ_k satisfies the (PS) condition over H_k for every $k \geq 1$ and either*

(a) *for every $k \geq 1$, φ_k satisfies the geometric assumption (7) of Lemma 3; or*

(b) *for every $k \geq 1$, φ_k satisfies the geometric assumption (8) of Lemma 4.*

Moreover, assume that the (non empty) set Z of critical points of φ_1 is compact and has the following property:

$$\text{(H)} \qquad \textit{if } x \in Z \textit{ and } m_1(x) = 0, \textit{ then } \nu(x) = 0.$$

Then there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, equation (2) has a kT-periodic solution which is not T-periodic.

PROOF. Let $\tilde{Z}_0 = \{x \in Z : m_1(x) = 0\}$ and $\tilde{Z}_1 = Z \setminus \tilde{Z}_0$. By (H), $\tilde{Z}_0$ is finite and $\tilde{Z}_1$ is compact. Hence, by Lemma 2, we may fix $k_0 \in \mathbb{N}$ such that (9) holds for every $x \in \tilde{Z}_1$ and $k \geq k_0$ (see Remark 1) and (10) holds for every $x \in \tilde{Z}_0$. Then Lemma 5 can be applied. □

REMARK 3. In case situation (b) holds with $d \geq 2$ and Z, Z_1 are both compact (hence Z_0 is also compact), we can drop assumption (H) in Theorem 1 since it follows from Lemma 2 that $m_k^*(x) < d$ for any $x \in Z_0$, $k \geq 1$ and then the above argument applies.

Using the same arguments together with Lemma 1, one can prove the following result for system (1), with $N \geq 1$.

THEOREM 2. *Assume that the functional φ_k $(k \geq 1)$, associated to system (1), satisfies the (PS) condition over H_k for every $k \geq 1$ and that either situation (a) or (b) of Theorem 1 holds. Moreover, assume that Z, the (nonempty) set of critical points of φ_1, is compact and*

$$\text{(H}'\text{)} \qquad m_1(x) \geq 1, \qquad \textit{for every } x \in Z.$$

Then there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ equation (1) has a kT-periodic solution which is not T-periodic.

PROOF. Simply note that now $Z = \tilde{Z}_1$, use Remark 1 and proceed as in Lemma 5. □

REMARK 4. It is easily seen that Z is compact in H_1 if and only if Z is bounded in $C([0,T];\mathbb{R}^N)$.

Next we give a sufficient condition for (H′) in Theorem 2 to hold.

LEMMA 6. *Assume that*

(H″) *$G(t,\cdot)$ is convex, for every $t \in [0,T]$ and there are no T-periodic solutions $x(t)$ of (1) such that $D_x G(t, x(t)) \equiv 0$.*

Then $m_1(x) \geq 1$ for any T-periodic solution $x(t)$ of (1).

PROOF. Let $x \in Z$ and denote by $\lambda_1(x)$ the first eigenvalue of (3), (4) with $k = 1$. It is well known that

$$\lambda_1(x) = \min\Big\{ \int_0^T [|\dot{y}(t)|^2 - (D_x^2 G(t, x(t)) y(t), y(t))]\, dt : y \in H_1, \|y\|_2 = 1 \Big\}.$$

Taking constant functions $y(t) \equiv y \in \mathbb{R}^N$ we get $\lambda_1(x) \leq 0$; and in fact $\lambda_1(x) < 0$ since otherwise $D_x^2 G(t, x(t)) \equiv 0$, which contradicts (H″). But $\lambda_1(x) < 0$ means precisely that $m_1(x) \geq 1$ and we are done. □

REMARK 5. It is clear that (H″) holds if $D_x^2 G(t, \cdot)$ is positive definite for every $t \in [0, T]$. Also, if an a priori bound $\|x\|_\infty < R$ for $x \in Z$ is known, we only require the strict convexity for $G(t, \cdot)$ on the ball $B_R(0)$ of $\mathbb{R}^N$.

3. Applications

In this section we apply the above theorems to a few special cases of equations (1) and (2).

THEOREM 3. *Consider equation (1) with $N \geq 1$, assume that $G(t, x)$ satisfies (H″) and*

- (i) *there exists a positive constant K such that $|\nabla G(t, x)| \leq K$ for every $(t, x) \in \mathbb{R} \times \mathbb{R}^N$;*
- (ii) *$\lim_{|x| \to \infty} \int_0^T G(t, x)\, dt = +\infty$.*

Then the conclusion of Theorem 2 holds true.

PROOF. For every $k \geq 1$ write $H_k = \mathbb{R}^N \oplus \tilde{H}_k$ (see section 1). From (i) (resp. (ii)) it follows that φ_k (resp: $-\varphi_k$) is coercive in $\tilde{H}_k$ (resp. :$\mathbb{R}^N$); thus we are in situation (b) of Theorem 2. In order to verify the (PS) condition, let $(x_n) \subseteq H_k$ be such that

$$\Big| \int_0^{kT} \big[\tfrac{1}{2} |\dot{x}_n(t)|^2 - G(t, x_n(t))\big]\, dt \Big| \leq M \tag{11}$$

$$\Big| \int_0^{kT} \big[(\dot{x}_n(t), \dot{y}_n(t)) - (\nabla G(t, x_n(t)), y(t))\big]\, dt \Big| \leq \epsilon_n \|y\|_{H_k} \tag{12}$$

for every $n \geq 1$, $y \in H_k$, where $M, \epsilon_n > 0$ and $\epsilon_n \to 0$. Taking $y = \tilde{x}_n$ in (12) and using (i) we get that $(\|\dot{x}_n\|_2)$ is bounded. Then $(\|\tilde{x}_n\|_\infty)$ and, from (11), $(|\int_0^{kT} G(t, x_n(t))\, dt|)$ are also bounded. From the convexity assumption we derive

$$G\Big(t, \frac{\bar{x}_n}{2}\Big) \leq \tfrac{1}{2} G(t, x_n(t)) + \tfrac{1}{2} G(t, -\tilde{x}_n(t)) \tag{13}$$

so that $\left(\int_0^{kT} G(t,\bar{x}_n/2)\,dt\right)$ is bounded and then from (ii), $(|\bar{x}_n|)$ is bounded. Hence $\left(\|x_n\|_{H_k}\right)$ is bounded and from classical arguments we can find a convergent subsequence (see e.g. [13]).

To end the proof it remains to show that any sequence (x_n) of T-periodic functions such that

$$\ddot{x}_n(t) + \nabla G(t, x_n(t)) = 0 \tag{14}$$

is bounded in H_1. Multiplying (14) by $\tilde{x}_n(t)$ and using (i) we get $(\|\dot{x}_n\|_2)$ bounded. Since G is convex, one has the inequality

$$G(t,y) \leq G(t,0) + (\nabla G(t,y), y) \quad \text{for } (t,y) \in \mathbb{R} \times \mathbb{R}^N,$$

and from (13) and (14) we get

$$\int_0^T G\left(t, \frac{\bar{x}_n}{2}\right) dt \leq \tfrac{1}{2} \int_0^T \left[|\dot{x}_n(t)|^2 + G(t,0) + G(t, -\tilde{x}_n(t))\right] dt. \tag{15}$$

The result then follows as before. □

THEOREM 4. *Consider equation (2), assume that $G(t,x)$ satisfies (H″),* (ii), *and*

(*iii*) $\lim_{|x|\to\infty}(G(t,x)/x^2)$ *uniformly in* t;

(*iv*) $\limsup_{|x|\to\infty}(g(t,x)/x) \leq M < (2\pi/T)^2$ *uniformly in* t.

Then the conclusion of Theorem 1 holds true.

PROOF. Writing $H_k = \mathbb{R} \oplus \tilde{H}_k$, conditions (ii) and (iii) show that we are again in situation (b) of Theorem 1. We only sketch the proof of the Palais-Smale condition which combines the arguments in [5] and [14]. Consider (11) and (12) above (where $\nabla G(t,x) = g(t,x)$). From (12), taking $y \equiv 1$, we get

$$\int_0^{kT} g(t, x_n(t))\,dt \to 0. \tag{16}$$

We claim that

$$\min_{0\leq t\leq kT} |x_n(t)| \leq c \tag{17}$$

for some constant $c > 0$ If not then, passing to a subsequence if necessary, we would have $\min|x_n(t)| \to +\infty$. Since (H″) implies that the function $\mathrm{sign}(x)g(t,x)$ is bounded from below, we get from (16) that $\left(\int_0^{kT} |g(t,x_n(t)|\,dt\right)$ is bounded (see[5]). Then from (12), taking $y = \tilde{x}_n(t)$, we get that $(\|\dot{x}_n\|_2)$ is bounded and from (11), $\left(\int_0^{kT} G(t,x_n(t))\,dt\right)$ is also bounded. Since $(\|\tilde{x}_n\|_\infty)$ is bounded, by taking into account (13) and (ii) we reach a contradiction.

Hence (17) holds. Let us prove now that $(|\bar{x}_n|)$ is bounded. If not, for a subsequence, $|\bar{x}_x| \to +\infty$ and we get from (11) and (iii) that $\tilde{x}_n(t)/\bar{x}_n \to 0$ in H_k (see[14]). But then $|x_n(t)| = |\bar{x}_n||1 + \tilde{x}_n(t)/\bar{x}_n| \to +\infty$ uniformly in t and this contradicts (17). Hence $(|\bar{x}_n|)$ is bounded and from (11) and (iii) $(\|\dot{x}_n\|_2)$ is also bounded.

Finally, let (x_n) be a sequence of T-periodic solutions of (2). A similar argument shows that (17) holds (use(15)). Now suppose that $|||x_n||| \equiv \|x_n\|_2 + \|\dot{x}_n\|_2 + \|\ddot{x}\|_2 \to +\infty$ (for some subsequence) and let $z_n(t) = x_n(t)/|||x_n|||$. We proceed as in [16] and use (iv) to write

$$g(t,x) = g_1(t,x) \cdot x + g_0(t,x), \tag{18}$$

where g_0, g_1 are functions such that $|g_0(t,x)| \leq c_1$ and $-c_2 \leq g_1(t,x) \leq c_3 < (2\pi/T)^2$ for every $(t,x) \in \mathbb{R}^2$ and some positive constants c_i, $i = 1,2,3$ (notice that, by (H″), $g(t,x)/x$ is bounded below for $|x|$ large). Dividing both members of (14) by $|||x_n|||$, taking limits and using (18) we see that $z_n(t) \to z(t)$ in $C^1([0,T];\mathbb{R})$, $z \not\equiv 0$ and for some function $\alpha(t)$ such that $-c_2 \leq \alpha(t) \leq c_3 < (2\pi/T)^2$ we have

$$\ddot{z}(t) + \alpha(t)z(t) = 0$$
$$z(0) - z(T) = 0 = \dot{z}(0) - \dot{z}(T)$$

From the Sturm-Liouville theory it follows that $z(t) \neq 0$ for every $t \in [0,T]$. Then $z_n(t)$ has a constant sign for n large enough and since $x_n(t) = |||x_n|||z_n(t)$ we have a contradiction with (17). This shows that $(|||x_n|||)$ is bounded and ends the proof of the theorem. □

EXAMPLE 1. Theorem 3 applies to $g(t,x) = \arctan x - h(t)$ where $h(t)$ is a continuous T-periodic function such that

$$-\frac{\pi}{2} < \frac{1}{T}\int_0^T h(t)\,dt < \frac{\pi}{2}.$$

EXAMPLE 2. Theorem 4 applies to

$$g(t,x) = \frac{x}{(1+x^2)^{1/4}} + h(t)$$

for any continuous T-periodic function.

REMARK 6. Let

$$g(t,x) = \frac{x}{(1+x^2)^{1/4}} + \frac{x}{1+x^2} + h(t),$$

where $h(t)$ is any continuous T-periodic function. Here all the assumptions of Theorem 4 are satisfied except for the convexity hypothesis (H″). Nevertheless we still have a priori bounds for T-periodic solutions of (2) and, for every $k \geq 1$, φ_k satisfies the (PS) condition over H_k; therefore Theorem 1 can be applied whenever $h(t)$ is such that condition (H) holds. Note that, for any $h(t)$, equation (2) has a T-periodic solution (this follows readily from the Saddle Point theorem) and thus condition (H) is of generic type with respect to $h(t)$ (see[10]).

REMARK 7. It is proved in [18, example 2] by means of phase-plane methods that equation (2) has (at least) a T-periodic solution if

$$g(t,x) = \frac{x + h(t)}{1 + x^2},$$

where $h(t)$ is a continuous T-periodic function. This follows also from the Saddle Point Theorem and again we may apply Theorem 1 whenever condition (H) holds (notice also that (H_0) holds for this case). This remark applies also to

$$g(t,x) = \frac{x}{1 + x^2} + h(t)$$

provided $\int_0^T h(t)\,dt = 0$.

We now study the Ambrosetti-Prodi type situation.

THEOREM 5. *Consider the following equation*

$$(P_s) \qquad x''(t) + g(t, x(t)) = s,$$

where $s \in \mathbb{R}$ is a parameter, $g \in C(\mathbb{R}^2; \mathbb{R})$ is T-periodic in its first variable and verifies the regularity assumptions of section 1. Suppose that

(i) $\lim_{|x| \to +\infty} g(t,x) = +\infty$ *uniformly in t;*

(ii) $g'_x(t, \cdot)$ *is strictly increasing for each $t \in \mathbb{R}$;*

(iii) $\lim_{x \to +\infty} g'_x(t,x) \le A < (2\pi/T)^2$ *for every $t \in \mathbb{R}$;*

(iv) $|G(t,x)| \le K_1 e^{-x} + K_2$ *for some positive constants K_1, K_2 and every $t \in \mathbb{R}$, $x \le 0$.*

Then there exists a constant $s_0 \in \mathbb{R}$ such that for every $s \ge s_0$ we can find $k(s) \in \mathbb{N}$ such that (P_s) has a kT-periodic solution which is not T-periodic, for any $k \ge k(s)$.

PROOF. Without loss of generality we may assume that $g(t,x)$ is positive. For the sake of clarity we divide the proof into several steps.

Step 1. It is known that under the sole assumption (i) there exists $s_1 \ge \max_{t \in \mathbb{R}} g(t, 0)$ such that (P_s) has zero, one or two T-periodic solutions according to whether $s < s_1$, $s = s_1$ or $s > s_1$, respectively (see [4]). Moreover, assumptions (ii) and (iii) imply that there are precisely one or two solutions for $s = s_1$ or $s > s_1$ respectively and, in the latter case, they are both non degenerate; this was proved in [12] for the dissipative Duffing equation with $A < (\pi/T)^2$, but it is immediately seen from the proof given there that this still holds true for our problem. In order to calculate the Morse index we shall obtain those solutions (for sufficiently large k) by using variational methods applied to the functional

$$\varphi_k(x) = \int_0^{kT} \left[\frac{|\dot{x}(t)|^2}{2} - g(t, x(t)) + s x(t) \right] dt,$$

where $x \in H_k$, as in [15].

Step 2 (minimization). For a given $s > K_1$ (see (iv)) consider the closed convex subset of H_1:

$$C = \{x \in H_1 : \|\dot{x}\|_2 \le 2sT^{3/2}, \bar{x} \le 0\}.$$

It is easily seen that φ_1 is coercive in C and, since it is a weakly lower semi continuous functional, we can find $u \in C$ such that

$$\varphi_1(u) \le \varphi_1(x) \tag{19}$$

for every $x \in C$. From (iv) we can estimate

$$\begin{aligned} \varphi_1(u) &\le \min\{\varphi(a), a \in]-\infty, 0[\,\} \\ &\le TK_2 + T\min\{K_1 e^{-a} + sa : a \in]-\infty, 0]\} \\ &= TK_2 + Ts(1 - \log(s/K_1)). \end{aligned} \tag{20}$$

On the other hand, it follows from (iii) that

$$G(t,x) \le A_1 \frac{x^2}{2} + A$$

for every $(t,x) \in \mathbb{R}^2$, where $A_2 > 0$, $0 < A_1 < (2\pi/T)^2$, and this implies that φ_1 is bounded below (in fact coercive) on $\tilde{H}$ by a constant which does not depend on s. From (20) we can thus find s_0 large enough so that for $s \ge s_0$ the function $u = u_s$ is such that

$$\bar{u} \ne 0 \tag{21}$$

From now on we fix s such that (21) holds. Choose $x = u \pm \epsilon$, ϵ small, in (19) and take limits to obtain

$$\int_0^T g(t, u(t))\,dt = sT.$$

Now choose $x = (1-\epsilon)\tilde{u} + \bar{u}$ in (19), $\epsilon > 0$ small, and take limits to get

$$\int_0^T |\dot{u}(t)|^2\,dt \le \int_0^T g(u(t))\tilde{u}(t)\,dt \le sT\|\tilde{u}\|_\infty \le sT^{3/2}\|\dot{u}\|_2$$

so that $\|\dot{u}\|_2 \le sT^{3/2} < 2sT^{3/2}$. Hence we conclude that u belongs to the interior of C and thus is an isolated local minimum for φ_1.

Step 3 (Mountain Pass Theorem). Let u be an isolated local minimum of φ_1. It is easily seen that φ_1 satisfies the (PS) condition over H_1; moreover,

$$\varphi_1(a_n) \to -\infty \text{ whenever } (a_n) \in \mathbb{R}, a_n \to +\infty$$

(these facts remain true for any φ_k, $k \geq 1$). Then, according to Lemma 3 (see Remark 2) we can find our second T-periodic solution v of (P_s) and we have

$$m_1(u) = 0 = \nu_1(u),$$
$$m_1(v) = 1, \nu_1(v) = 0.$$

According to lemmas 1 and 2 we can fix $k(s)$ so large that

$$m_k(u) = 0 = \nu_k(u), \tag{22}$$
$$m_k(v) \geq 2 \tag{23}$$

for any $k \geq k(s)$.

Step 4. Take any $k \geq k(s)$ and assume by contradiction that u, v are the only kT-periodic solutions of (P_s) (in particular they are isolated in H_k). For each $n \geq 1$ consider the critical groups $C_n(\varphi_k, u)$ (see[9]). Since $\nu_k(u) = 0$ we have

$$\dim C_n(\varphi_k, u) = \delta_{n,m_k(u)},$$

where δ stands for the Kronecker symbol ([9; corollary 8.3]). From (22) we get $C_0(\varphi_k, u) \neq 0$ and by [9, Theorem 8.6] u is a local minimum for φ_k. But then we can apply Lemma 3 to φ_k (see Step 3) in order to get a second solution — which is precisely v — such that $m_k(v) \leq 1$. This contradicts (23) and ends the proof of the theorem. □

EXAMPLE 3. Let $a(t), h(t)$ be continuous functions with minimal period T, $a(t) > 0$, and λ be a positive constant with $\lambda < (2\pi/T)^2$. Then if $\frac{1}{T}\int_0^T h(t)\,dt$ is sufficiently large, the equation

$$x''(t) + a(t)e^{-x(t)} + \lambda x(t) = h(t)$$

admits infinitely many subharmonics with minimal period kT, k prime.

Our last theorem extends partially corollary 8 in [18].

THEOREM 6. *Consider equation (2) where $g(t,x)$ is T-periodic in its first variable and satisfies the regularity assumptions of section 1. Suppose that $g(t,x)$ is bounded below, $g'_x(t,x) > 0$ and there exists a positive constant r_1 such that*

$$\operatorname{sign}(x)\int_0^T g(t,x)\,dt > 0 \tag{24}$$

for every $t \in \mathbb{R}$, $|x| \geq r_1$. Then the conclusion of Theorem 1 holds true.

PROOF. Again, we divide the proof in several steps and use an argument similar to the one in Theorem 5.

Step 1 (a priori bounds). For a given $k \geq 1$ let $x(t)$ be a $\tau \equiv kT$-periodic solution of (2) and $m > 0$ be such that $-m < \min_{\mathbb{R}^2} g$. Since $\int_0^\tau g(t, x(t))\,dt = 0$, we have $|x(t_0)| < r_1$ for some $t_0 \in [0, \tau]$. Multiplying (2) by $\tilde{x}(t)$ we get

$$\|\dot{x}\|_2 \leq \tau^{3/2} m \equiv c. \tag{25}$$

Hence we have the a priori bound

$$\|x\|_\infty < r_1 + \tau^2 m \equiv r$$

for every τ-periodic solution of (2). Next we make the following remark: if $f(t, x)$ is a function such that $f(t, x) \geq g(t, r_1)$ for every (t, x) in $\mathbb{R} \times [r_1, +\infty[$ and $f \geq -m$ in $\mathbb{R}^2$, we still have the bound (25) and $\min x(t) < r_1$ for every τ-periodic solution $x(t)$ of

$$x''(t) + f(t, x(t)) = 0 \tag{26}$$

Since the set Z of T-periodic solutions of (2) is a priori bounded and $m_1(x) \geq 1$ for every $x \in Z$, we can choose $k_0 \in \mathbb{N}$ such that

$$m_k(x) \geq 2$$

for every $x \in \mathbb{Z}$ and $k \geq k_0$ (see Remark 1 and Remark 4). In the sequel we fix $k \geq k_0$ and assume by contradiction that all kT-periodic solutions of (2) are T-periodic. We shall denote $\tau = kT$.

Step 2 (truncation). Let $\lambda > 0$ be a small positive number to be chosen below and consider the function

$$f(t,x) = \begin{cases} g'_x(t,r)(x-r) + g(t,r), & x \geq r; \\ g(t,x), & -r \leq x \leq r; \\ \theta_\lambda(t,x), & -2r \leq x \leq r; \\ \theta_\lambda(t,-2r) - \lambda(x+2r), & x \leq -2r, \end{cases}$$

where θ_λ is such that $f \in C(\mathbb{R}^2; \mathbb{R})$ is T-periodic in its first variable, its first derivative with respect to its second variable is continuous and $-m \leq \theta_\lambda(t, x) \leq g(t, -r)$.

Suppose that λ verifies:

$$0 < \lambda < (c\tau^{3/2})^{-1}\Big(-\int_0^\tau g(t,-r)\,dt\Big). \tag{27}$$

Now we claim that if $u(t)$ is a τ-periodic solution of (26), and $u(t_1) \equiv \max u(t)$, then either

(a) $u(t_1) < -2r$; or

(b) $u(t_1) > r_1$.

Indeed, suppose to the contrary that $-2r \le u(t_1) \le -r_1$. By the remark made in Step 1, we know that $\|\dot{u}\|_2 < c$; then we have

$$-2r - c\tau^{1/2} \le u(t) \le -r_1, \tag{28}$$

for every $t \in [0, \tau]$. But our choice of λ implies that

$$\int_0^\tau (\theta_\lambda(t,x) - \lambda(x+2r))\,dt < 0$$

for any $-2r - c\tau^{1/2} \le x \le -2r$, so that we have $\int_0^\tau f(t,x)\,dt < 0$ in $[-2r - c\tau^{1/2}, -r_1]$. Since $\int_0^\tau f(t,u(t))\,dt = 0$, we get a contradiction with (28). This proves the claim.

Now (a) means that $u(t)$ is the unique τ-periodic (in fact, T-periodic) solution of

$$\ddot{u} - \lambda u(t) = \theta_\lambda(t,-2r) + \lambda 2r. \tag{29}$$

On the other hand if (b) holds, and according to the remark in Step 1, we have $\|u\|_\infty < r$ and $u(t)$ is a τ-periodic solution of (2). Hence we conclude that if $u(t)$ is a τ-periodic solution of (26) verifying (a) then any other τ-periodic solution of (26) must be a solution of (2).

Step 3 (the modified problem). Setting $F(t,x) = \int_0^x f(t,x)\,ds$, consider the functional

$$\Psi(x) = \int_0^\tau \Big[\frac{\dot{x}^2(t)}{2} - F(t,x(t))\Big]\,dt,$$

defined on H_k. Since the function $f(t,x)$ is coercive we may proceed as in Step 2 in the proof of Theorem 5 and choose

$$u = u_\lambda(t) \in C \equiv \{x(t) \in H_k : \|\dot{x}\|_2 \le 2c, \bar{x} \le 0\}$$

such that

$$\Psi(u) \le \Psi(x) \qquad \text{for every } x \in C.$$

Now, denoting

$$c(r) = 4r \max\{|g(t,x)|,\ t \in \mathbb{R},\ -r \le x \le 0\},$$
$$\bar{g} = \frac{1}{T}\int_0^T g(t,-r)\,dt < 0,$$

we have

$$-F(t,x) \le c(r) - xg(t,-r) + \lambda\tfrac{1}{2}(x+2r)^2$$

for every $t \in \mathbb{R}$, $x \leq -2r$, so that

$$\begin{aligned}\Psi(u) &\leq \min\{\Psi(a) :\in]-\infty, 0]\} \\ &\leq \tau c(r) + \tau \min\Big\{\frac{\lambda}{2}(a+2r)^2 - \bar{g}a : a \in]-\infty, 0]\Big\} \\ &= \tau c(r) + \tau\Big(2r\bar{g} - \frac{\bar{g}^2}{2\lambda}\Big),\end{aligned}$$

and this last expression tends to $-\infty$ as $\lambda \to 0$. Since $\|\dot{u}\|_2 \leq 2c$ we may choose λ so small that ((27) holds and)

$$\bar{u} = \bar{u}_\lambda < -2r - 2c\tau^{1/2}. \tag{30}$$

In particular we have $\bar{u} < 0$ and we can proceed as in Step 2 of the proof of Theorem 5 in order to prove that u is a local minimum of Ψ and hence a τ-periodic solution of (26). Moreover from (30) we see that situation (a) above holds and from (29) we get that u is non degenerate (in particular, u is isolated in H_k).

Since we found an isolated local minimum u for Ψ and, as it is easily seen, Ψ satisfies the (PS) condition over H_k, we have the geometric setting of Lemma 3. According to our previous remarks we have $m_\Psi(u) = m^*_\Psi(u) = 0$ and $m_\Psi(v) \geq 2$ for every critical point $v \neq u$ of Ψ. It follows from Lemma 5 applied to Ψ that equation (26) admits a kT-periodic solution v which is not T-periodic. Necessarily $v \neq u$ and v is a kT-periodic solution of (2), contrary to our assumption. □

COROLLARY. *Consider equation*

$$x''(t) + g(x(t)) = h(t), \tag{31}$$

where $g \in C^1(\mathbb{R};\mathbb{R})$ is bounded below, $g'(x) > 0$ and $h(t)$ is a continuous function with minimal period T. If

$$\bar{h} \in \text{range}(g), \tag{32}$$

equation (31) admits subharmonic solutions with minimal period kT, for every k large and prime.

REMARK 8. Let us notice that condition (32) is also necessary for the existence of a subharmonic solution of (31). Moreover, setting $m = \min_{\mathbb{R}} g$ and being $r_1 > 0$ such that $\operatorname{sign} x(g(x) - \bar{h}) > 0$ for $|x| \geq r_1$ we have the a priori bound

$$\|x\|_\infty < r_1 + \sqrt{T}\left(2\int_0^T |h(t)|\,dt - Tm\right) \equiv R$$

for every T-periodic solution of (31). According to Remark 5 (see also the proof given above) it is sufficient to assume a strict monotonicity on the interval $[-R, R] \subseteq \mathbb{R}$.

EXAMPLE 4. In [18; Example 3] it is shown that equation $\ddot{x}(t) + e^{x(t)} = h(t)$ has infinitely many subharmonics provided that $h(t) > 0$. The above corollary asserts that it is sufficient to have $\bar{h} > 0$.

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