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BORSUK'S INFLUENCE ON INFINITE-DIMENSIONAL TOPOLOGY

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Dedicated to the memory of Karol Borsuk

Karol Borsuk had a great and continuing influence on the development of Infinite-dimensional Topology. It is derived from three sources. Firstly, his mathematical work produced fundamental ideas which permeated the entire fabric of topology and are of particular importance in the infinite dimensional case. He explored basic relationships and provided a conceptual framework and technical apparatus which stimulated, organized, and channelled further work. Secondly, he had an incalculable influence with a few audacious conjectures concerning the topology of the Hilbert cube and Absolute Neighborhood Retracts (ANR's). Thirdly, by his strong support of younger mathematicians, unfailing hospitality, and continuous efforts on behalf of international scientific communication and cooperation, he worked to create the kind of environment, in Poland as well as internationally, in which mathematics flourishes most strongly.

In his work, Borsuk was almost never interested in infinite-dimensionality *per se*, but from the very beginning of his career he freely made use of infinite-dimensional objects as part of the natural range of his geometric and topological thought. He proved quite a few theorems involving infinite-dimensional objects in an essential way. These include the characterization [B1] of unicoherence by the connectivity of the space of functions into the circle and one of the foundational papers [B8] on the homology of mapping spaces, an important topic in contemporary topology. The entire series of papers on maps into spheres and his cohomotopy groups is filled with fundamental concepts. There is also the proof [B6] that the space of self-maps of the closed interval is homeomorphic with its subspace of n-times continuously differentiable functions. The sequence [B2], [B3], and [B10] treating hyperspaces of closed subsets is full of influential ideas: in [B2], with Ulam, he introduced the n-fold symmetric product, now used computationally; in [B3], with Mazurkiewicz, he proved that the hyperspace of non-void closed subsets of a metric continuum is arcwise connected, under the Hausdorff metric; and in the others he altered the Hausdorff metric so that the subspace of embedded ANR's is complete and investigated its properties. The paper with Mazurkiewicz was particularly influential in that it stimulated Wojdysławski to prove [Wo] that if the continuum is locally connected, then the hyperspace is an Absolute Retract (AR) and to conjecture that it is in fact homeomorphic with the Hilbert cube. Wojdysławski's Conjecture, initially in its concrete form for the hyperspace of the unit interval, concerned an object so radically different from the convex subsets previously considered that it generated an intense research effort in the late 1960's and early 1970's. (It was finally proven in 1972 by D. W. Curtis and R. M. Schori [CuS] (cf. [T1]). The new metric, however, appears to have suffered undeserved neglect for many years, as Ferry and Nofech have noted that it provides a naturally defined classifying space for compact ANR fibrations over finite-dimensional bases. Recently, under the lead of M. Gromov, these ideas have become of considerable use in differential geometry. (Cf. [Gr] and [GrP].)

There is also a lone paper on Hilbert and Hilbert cube manifolds, in which Borsuk defines the latter concept and makes two sweeping conjectures of the greatest importance, to be discussed below.

Finally, there is the famous and important pair [B5] and [B7]. In [B5], Borsuk defined ANR's and showed that his concept of local contractibility, introduced in [B7], characterizes the ANR's among the finite-dimensional, metric compacta. In [B7], he complements this result by giving an infinite-dimensional, locally contractible, compact, metric space that is not an ANR. The problem of giving a tractable characterization of infinite-dimensional ANR's is still open (cf. Geoghegan's contribution to the problem set of [BoK]).

Influential as his theorems about explicitly infinite-dimensional objects were, by far the most important portions of Borsuk's work for Infinite-Dimensional Topology were the Theory of Retracts and the Theory of Shape. Throughout the development of Infinite-Dimensional Topology a constantly recurring leitmotif, and perhaps the central theme, has been the interplay between manifolds and ANR's. The theories of infinite-dimensional manifolds of all sorts are now viewed as simultaneous stable theories of ANR's, polyhedra, CW-complexes, and n-manifolds (cf. [We1]). One of the more active areas of research, historically, has been simply to discover which of the many naturally occurring infinite-dimensional spaces are manifolds. Today, in the wake of H. Toruńczyk's axiomatic characterization of Hilbert and Hilbert cube manifolds, these problems are mostly those of detecting ANR's. It is quite remarkable that many of the most important problems may still be so formulated. (See the problems in [C3], [G], [BoK], [We2].) The theory of ANR's has provided a major part of the conceptual framework within which this work developed, as well as a rich collection of examples and constructive techniques. Many of these were due to Borsuk.

The influence of Shape Theory, on the other hand, has been of a different character. Whereas the theory of Retracts was already mature when Infinite-Dimensional Topology began to draw on it heavily, Shape Theory was inaugurated by Borsuk in his address at the Infinite-Dimensional Topology Symposium at Louisiana State University in 1967 [A2]. This talk made quite an impression on some of us in the audience, and I still remember the excitement with which I, then a graduate student, listened. Shape Theory was set, for convenience, in Hilbert space and addresses problems of concern to Infinite-Dimensional Topology. The methodology is even more relevant, especially to non-compact Hilbert cube manifolds, where problems at the ends require similar techniques. The result was that as Shape Theory and Infinite-Dimensional Topology matured together, there was considerable exchange of ideas and influence, and several people have made major contributions to both fields, i.e., Chapman, Ferry, and Geoghegan. An outstanding example of the interaction of the two fields is the following: Chapman [C1] proved that the Shape Theory of metric compacta is dual to a weak proper homotopy theory of a certain class of Hilbert cube manifolds and that Shape isomorphisms are dual to homeomorphisms. This interested the Shape theorists considerably. Later, D. A. Edwards and H. Hastings [EH] defined Strong Shape Theory by reversing the duality but using the proper homotopy theory of those manifolds. Strong Shape is now a significant part of the field. Other examples of fruitful interplay that have led to significant advances include the question of finiteness obstructions (cf. [EG], [F], a paper of S. Ferry in [MS]), pointed versus unpointed questions (cf. [Dy], [BrG] and an article of H. Hastings and A. Heller in [MS]), approximate and shape fibrations (cf. [L], [MR], [C4], [Q], [Hu], and a paper of D.S. Coram in [MS]), and the question of the existence of cell-like, dimension-raising maps [W], [D].

Perhaps even more important to Infinite-Dimensional Topology than his work has been the result of a few penetrating questions Borsuk posed. In 1938 he contributed to the Scottish Book [M] a daring two-part question about the Hilbert cube (number 175):

(a) "Is the product with the Hilbert cube of a triod (three arcs sharing a common end point) homeomorphic with the Hilbert cube?"

(b) "Is the product of infinitely many triods a Hilbert cube?"

In 1961, in [B11], Borsuk extended the Scottish Book Questions to the limit:

(1) (*Stabilization*) "Is every product with the Hilbert cube of a polyhedron, or, indeed a compact ANR, a Hilbert cube manifold?" and
(2) (*Triangulation*) "Is every Hilbert cube manifold homeomorphic with the product with the Hilbert cube of a polyhedron?"

The boldness of these questions can only be appreciated by reading [B11]. (Recall, this is the paper in which Hilbert cube manifolds were first defined. There had been no progress on the Scottish Book problem or Wojdysławski's Conjecture in the two decades since their enunciation.)

These four questions, together with the one on the homotopy type of compact ANR's from his 1954 ICM Address and their unstated counterparts for Hilbert manifolds, had a decisive effect. In 1963, R. D. Anderson solved the Scottish Book question positively; perhaps the most important result was Anderson's permanent interest in Infinite-Dimensional Topology. He quickly used the Hilbert cube to complete work of Cz. Bessaga, Kadec, and A. Pełczyński on the topological classification of Frechet spaces and then orchestrated an intensive thirteen-year effort which led, almost as if by design, one year after R. Edwards proved the Stabilization Conjecture, to the astonishingly elegant and powerful topological characterization of Hilbert cube manifolds by Toruńczyk in 1976: A locally compact ANR is a Hilbert cube manifold if and only if each pair of maps of the Hilbert cube into it may be approximated by a pair of maps with disjoint images. Toruńczyk later gave a similar characterization of Hilbert manifolds, and there are now quite a few characterizations along this line. It is worthy of note, a propos of unexpected dividends from well-chosen questions and of infinite-dimensional manifolds as stabilized polyhedra, that Chapman's proof of Borsuk's Triangulation Conjecture led immediately to his proof of the topological invariance of Whitehead torsion, a long sought-after conjecture of finite dimensional topology. All this development was in some sense set in motion by Borsuk. (Further discussion of these events, not suitable for this note, may be found in Toruńczyk's commentary on problem 175 in the Scottish Book [M] and also in [A1], [BP], [C2], [C3], [T1], [T2], and the article of J. Walsh in [MS]).

Finally, it is important to stress that this mathematical development did not happen in a human relations vacuum. A significant part of Borsuk's mathematical impact flowed from his unique personality. I find I cannot describe it and must leave that to others who knew him better. But an inkling of his attitude is given by the phrase "he treasured mathematical life and those engaged in it." He helped and supported many younger Polish topologists in diverse ways. However, during the past twenty-five years there has been a quite remarkable scientific contact between Polish topologists and functional analysts and American mathematicians of similar interests; the rapid development of Infinite-Dimensional Topology in the 1960's and 1970's is inconceivable without it. Borsuk was actively involved in these exchanges. He also welcomed foreign mathematicians to his seminar at the Mathematical Institute of the Polish Academy of Sciences in Warsaw with a graciousness and courtesy that put them immediately at ease, and he was always pleased to share his time and exchange insights. On a more private level, Borsuk and his wife, Zofia, extended a warm and delightful hospitality that greatly enchanced the cordial scientific atmosphere. In my opinion, this hospitality was a significant contribution to the success of the international cooperation.

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