Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 1, 1993, 15–33

SPHERES AND SYMMETRY: BORSUK'S ANTIPODAL THEOREM

H. STEINLEIN

Dedicated to the memory of Karol Borsuk (1905–1982)

Last century, the mathematicians left the area of normal intuition when they started to consider not only Euclidean spaces of dimensions 1, 2, and 3, but of any finite dimension. By this, they won a huge amount of new objects, among which the spheres S^n belong to the simplest and most important ones.

It was very encouraging that even these seemingly simple objects turned out to have highly interesting topological properties. Some of the basic facts were found during the first decades of this century:

a) From the Brouwer fixed point theorem ([18]; Poincaré [61] and Bohl [12] already knew equivalent results) it followed that the spheres S^n are not contractible (cf. Borsuk [13, p. 162]).
b) Hopf [37] gave a complete description of the homotopy classes of maps $f : S^n \to S^n$ by relating these to the Brouwer degree.
c) Moreover, Hopf [38] gave examples for the surprising fact that there are essential maps $f : S^n \to S^m$ with $n > m$, i.e., maps which cannot be continuously deformed to a constant map. This started the highly complicated theory of homotopy groups of spheres.

Borsuk's antipodal theorem introduced a new concept to these considerations: symmetry. Spaces are now considered as topological spaces with some symmetry, e.g. the antipodal symmetry on spheres; maps between such spaces should respect the symmetry.

There are good reasons to take symmetry into account: Symmetries appear naturally in many situations, not only the antipodal symmetry on spheres, and, as the antipodal theorem shows, they add a lot of information to the purely topological setting.

So it is not surprising that Borsuk's original papers [14, 15] on the antipodal theorem had an enormous influence on mathematical research. A deep theory evolved from his results, including a large number of applications and a broad variety of generalizations.

The tenth anniversary of Borsuk's death on January 24, 1982 together with the sixtieth anniversary of Borsuk's announcement of his "Antipodensatz" in March 1932 are good opportunities to honour this outstanding mathematician with a survey article which describes his antipodal theorem and its relations to different fields of mathematics as well as the major aspects of the theory that evolved from this fundamental idea.

In contrast to [68], this new survey paper does not aim at any completeness. It is intended to give some insight in the classical result and the ideas of its proofs, to point out the major lines and limitations of generalizations, and last but not least give a short impression of the large variety of applications.

From this it should be clear that I will not present the theory in its utmost generality. In particular, I will describe the theory only in the framework of finite–dimensional spheres (except of the section 5 on index theories, where this restriction wouldn't make sense). In many cases, the generalization of Borsuk's theory to boundaries of symmetric neighborhoods of the origin in infinite–dimensional spaces is easy to obtain, once the appropriate (usually nontrivial) generalization of the Brouwer degree is available. Even though this kind of generalizations is of great interest for applications (cf. section 6), I will omit them, because they would provide no additional insight in the topology behind this theory.

1. Basic Facts and Tools

We recall a few notions which are essential for the understanding of this paper (throughout this paper, all maps between topological spaces are assumed to be continuous).

Brouwer Degree. Let

$$\mathcal{L} := \{(f, A) : A \subset \mathbb{R}^n \text{ compact}, \ f : A \to \mathbb{R}^n, 0 \notin f(\partial A)\}.$$

The Brouwer degree $\deg : \mathcal{L} \to \mathbb{Z}$ has in particular the following properties:

(i) (*Normalization*) If $f \in C^1(\mathring{A})$ and

$$\det Df(x) \neq 0, \qquad \text{for all } x \in f^{-1}(0),$$

then

$$\deg(f, A) = \sum_{x \in f^{-1}(0)} \operatorname{sign} \det Df(x).$$

(ii) (*Homotopy*) If $F : A \times [0,1] \to \mathbb{R}^n$ is a map with $0 \notin F(\partial A \times [0,1])$, then

$$\deg(F(\cdot, 0), A) = \deg(F(\cdot, 1), A).$$

(iii) (*Solution*)

$$\deg(f, A) \neq 0 \Longrightarrow f^{-1}(0) \neq \emptyset.$$

Let $f : S^{n-1} \to S^{n-1}$, and let

$$F : K^n := \{x \in \mathbb{R}^n : |x| \leq 1\} \longrightarrow \mathbb{R}^n$$

be an extension of f. Then we define the degree

$$\deg f := \deg(F, K^n).$$

It is an easy consequence of the homotopy property that $\deg f$ is well–defined, i.e., independent of the particular F.

A map $f : S^{n-1} \to S^{n-1}$ is called *essential* if it is not homotopic to a constant map. It is well known that

$$f : S^{n-1} \to S^{n-1} \text{ essential} \Longleftrightarrow \deg f \neq 0.$$

Cohomology. Let (X, A) be a topological pair, i.e., X is a topological space and $A \subset X$. Let K be a field.

Cohomology theories assign to each pair (X, A) and each $k \in \mathbb{Z}$ a vector space

$$H^k(X, A) = H^k(X, A; K)$$

(we will write $H^k(X) := H^k(X, \emptyset)$ for short) such that at least the following properties are given:

(i) (*Homomorphism*) A map $f : (X, A) \to (Y, B)$ (i.e., $f : X \to Y$ with $f(A) \subset B$) induces homomorphisms

$$f^* : H^k(Y, B) \longrightarrow H^k(X, A).$$

(ii) (*Constant Maps*) $f^* = 0$ for all constant maps f and all $k \neq 0$.

(iii) (*Homotopy*) If $f \sim g : (X, A) \to (Y, B)$ (i.e., there is an $F : X \times [0,1] \to Y$ with $F(A \times [0,1]) \subset B$ and $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$), then $f^* = g^*$.

(iv) (*Exactness*) There is an exact sequence

$$\cdots \longrightarrow H^k(X, A) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(A) \longrightarrow H^{k+1}(X, A) \longrightarrow \cdots$$

with the inclusions $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$.

(v) (*Cup Product*) Let $V_1, V_2 \subset X$ be open. There is a product

$$\cup : H^j(X, V_1) \times H^k(X, V_2) \longrightarrow H^{j+k}(X, V_1 \cup V_2)$$

($j, k \in \mathbb{N}_0$) such that, if $f : (X; V_1, V_2) \to (Y; W_1, W_2)$ (i.e., $f : X \to Y$ with $f(V_i) \subset W_i$, $i = 1, 2$), then for $\alpha \in H^j(Y, W_1)$ and $\beta \in H^k(Y, W_2)$

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta \in H^{j+k}(X, V_1 \cup V_2)\,.$$

In general, properties (i)–(v) do not uniquely determine the cohomology theory, but for us there is no reason why to be more explicit.

The following example will be of particular interest. Let $\mathbb{R}P^n$ be the n-dimensional real projective space, i.e., the sphere S^n with antipodal points identified. Then

$$H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & \text{for } k = 0, \dots, n; \\ 0, & \text{for all other } k\text{'s}. \end{cases}$$

Let $\alpha_k \in H^k(\mathbb{R}P^n; \mathbb{Z}/2)$ be the unique nontrivial element ($k = 1, \dots, n$). Then

$$\alpha_k = \alpha_1^k = \alpha_1 \cup \cdots \cup \alpha_1 \qquad (k\text{-times}).$$

Lusternik-Schnirelmann Category. Let A be a closed subset of X. The Lusternik-Schnirelmann category of A in X is defined by

$$\operatorname{cat}_X A := \min\big\{k \in \mathbb{N}_0 : \exists \text{ closed } A_1, \dots, A_k \subset A : A_1 \cup \cdots \cup A_k = A, \\ \text{and all } A_j \text{ are contractible in } X\big\}.$$

Joins. Let $n \in \mathbb{N}$, and let

$$\Gamma_n := \Big\{(t_1, \dots, t_n) \in [0,1]^n : \sum_{i=1}^n t_i = 1\Big\}.$$

For the sets $X_1, \dots, X_n$ we define the join

$$X_1 * \cdots * X_n := (\Gamma_n \times X_1 \times \cdots \times X_n)/\sim\,,$$

where

$$(s_1, \dots, s_n, x_1, \dots, x_n) \sim (t_1, \dots, t_n, y_1, \dots, y_n) \\ \iff s_1 = t_1 \wedge \cdots \wedge s_n = t_n \wedge (t_i \neq 0 \Rightarrow x_i = y_i).$$

We will use this definition only for compact spaces $X_1, \dots, X_n$, in which case there is a natural topology on $X_1 * \cdots * X_n$.

2. The Classical Antipodal Theorem

The antipodal theorem has many fathers. The first variants are due to L. Schnirelmann [64] together with L. Lusternik [52, 53, 54]. Another version (cf. Theorem B below) was conjectured by S. Ulam (cf. [15, p. 178]), seemingly his only contribution to the theory. But there is no doubt that Borsuk's classical paper [15] was the real starting point for the theory.

An important sign for the structural profundity of the antipodal theorem is the large variety of equivalent formulations of this result. The following selection of eight versions contains the most important ones and those which we will need later.

A. *Each odd* $f : S^n \to S^n$ *is essential* (*more precisely* $\deg f$ *is odd*) [15].

B. *For each* $f : S^n \to \mathbb{R}^n$ *there exists an* $x \in S^n$ *with* $f(x) = f(-x)$ [15].

C. *Each closed cover* $\{A_1, \ldots, A_{n+1}\}$ *of* S^n *contains at least one set* A_i *with* $A_i \cap (-A_i) \neq \emptyset$ [15, 53].

D. *Let* $A_1, \ldots, A_{n+1} \subset S^n$ *be closed with* $A_i \cap (-A_i) = \emptyset$ *for* $i = 1, \ldots, n+1$. *If*
$$\bigcup_{i=1}^{n+1} A_i \cup (-A_i) = S^n,$$
then $A_1 \cap \cdots \cap A_{n+1} \neq \emptyset$.

E. *Any* $f : K^{n+1} \to \mathbb{R}^{n+1}$ *with an odd restriction* $f\,|_{S^n}$ *has a zero* (*or equivalently a fixed point*).

F. *For* $m < n$, *there is no odd* $f : S^n \to S^m$.

G. $\mathrm{cat}_{\mathbb{R}P^n} \mathbb{R}P^n \geq n + 1$ (*in fact, equality holds*) [52, 53, 54, 64].

H. *Let* M, N *be linear subspaces of a normed space* E *with* $\dim M < \dim N$. *Then there is an* $x_0 \in N \backslash \{0\}$ *with* $\mathrm{dist}(x_0, M) = \|x_0\|$, [48].

REMARKS. Theorems A–C are the original antipodal theorems of Borsuk. Not all of the equivalence proofs are obvious. The papers [7, 19, 26, 62] should cover most of the nontrivial parts. For other equivalent formulations, cf. e.g. [30, 73, 75].

In order to illustrate the topological background of these results, I will sketch in the following some proofs of the antipodal theorem.

Degree Theory. Extend the odd $f : S^n \to S^n$ in Theorem A to an odd $\bar{f} : K^{n+1} \to \mathbb{R}^{n+1}$. It is possible to approximate $\bar{f}$ sufficiently close by a smooth odd $\tilde{f} : K^{n+1} \to \mathbb{R}^{n+1}$ with $D\tilde{f}(x)$ nonsingular for all $x \in \tilde{f}^{-1}(0)$. Here the approximation by a smooth map f_0 is standard, whereas the further approximation in order to fulfill the nonsingularity condition is based on a nontrivial use of a transversality argument (Sard's lemma), cf. e.g. [34]. Since $0 \in \tilde{f}^{-1}(0)$ and with $x \in \tilde{f}^{-1}(0)$ also $-x \in \tilde{f}^{-1}(0)$, $\tilde{f}$ has an odd number of zeros, thus from normalization and homotopy

$$\deg f = \deg(\tilde{f}, K^{n+1}) \equiv 1 \pmod 2.$$

Cohomology. Assume that we have a closed covering of $\mathbb{R}P^n$ by sets $W_1, \ldots, W_m$, which are contractible in $\mathbb{R}P^n$ to a point, and let $V_1, \ldots, V_m$ be open contractible neighborhoods. Then with $i_k : V_k \hookrightarrow \mathbb{R}P^n$

$$i_k^* = 0 : H^1(\mathbb{R}P^n) \to H^1(V_k).$$

By exactness,

$$j_k^* : H^1(\mathbb{R}P^n, V_k) \longrightarrow H^1(\mathbb{R}P^n)$$

is onto. Let $\beta_k \in j_k^{*\,-1}(\alpha_1)$, where $\alpha_1 \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ is the unique nontrivial element. Then

$$\beta_1 \cup \cdots \cup \beta_m \in H^m(\mathbb{R}P^n, V_1 \cup \cdots \cup V_m) = H^m(\mathbb{R}P^n, \mathbb{R}P^n) = 0,$$

and hence

$$\alpha_1^m = j_1^*\beta_1 \cup \cdots \cup j_m^*\beta_m = j^*(\beta_1 \cup \cdots \cup \beta_m) = 0,$$

i.e., $m \geq n+1$.

Combinatorics. The unit sphere

$$\hat{S}^n \text{ in } (\mathbb{R}^n, \|\cdot\|_1), \qquad \text{where } \|x\|_1 := \sum_{i=1}^{n} |x_i|,$$

is symmetrically homeomorphic to the ordinary sphere S^n. $\hat{S}^n$ has a canonical symmetric triangulation L with vertices

$$\pm e^{(i)} := \pm(\delta_{i\,1}, \ldots, \delta_{i\,n+1}) \qquad (i = 1, \ldots, n+1).$$

In order to prove Theorem D, we choose a symmetric subdivision K of L such that, for all $k \in \{1, \ldots, n+1\}$, each simplex S in K does not intersect with both A_k and $-A_k$, and such that

$$S \cap A_k \neq \emptyset, \qquad \text{for } k = 1, \ldots, n+1,$$

implies $A_1 \cap \cdots \cap A_{n+1} \neq \emptyset$ (i.e., to say diam S is smaller than the Lebesgue number of the cover $\{A_1, \ldots, A_{n+1}, -A_1, \ldots, -A_{n+1}\}$).

For a vertex $p \in K$, let

$$j_p := \min\{k \in \{1, \ldots, n+1\} \,:\, p \in A_k \cup (-A_k)\}.$$

Then the map $\tilde{f} : K^0 \to L^0$ between the 0-skeletons K^0 and L^0 of K and L,

$$\tilde{f}(p) := \begin{cases} (-1)^{j_p} e^{(j_p)}, & \text{for } p \in A_{j_p}; \\ (-1)^{j_p+1} e^{(j_p)}, & \text{for } p \in (-A_{j_p}), \end{cases}$$

induces a simplicial map $f : K \to L$. From a combinatorial lemma, which goes back to Tucker (cf. [26, 30, 33, 49, 71]), we can deduce that an odd number of simplices in K is mapped onto the simplex $[e_1, -e_2, \dots, (-1)^n e_{n+1}]$.

Other Proofs. An algebraic proof is given in [5, 46]. Of practical interest are proofs of the Borsuk-Ulam result based on a homotopy extension argument [2, 10, 58, 70], since this provides a framework for numerical techniques. Other interesting proofs may be found e.g. in [25, 74, 76].

3. The Role of Symmetry

In this section, we try to understand better the role of the antipodal condition in the different variants of Borsuk's theorem.

In the thirties and forties, it was mainly H. Hopf who tried to get a deeper insight in this symmetry condition by replacing it by closely related conditions, which nevertheless are of a somewhat different nature. They are good examples to make clear in what sense the antipodal condition is essential. They will lead us to the next section, in which we will consider symmetries with respect to group actions which turn out to yield the most appropriate framework for this theory.

A. H. Hopf [39] introduced the following notions.

DEFINITION 3.1. (i) A map $f : S^n \to \mathbb{R}^n$ is called *free* if there exists an $h : S^n \to S^n$ such that

$$f(x) \neq f(h(x)), \qquad \text{for all } x \in S^n.$$

(ii) A closed covering $\{A_1, \dots, A_m\}$ of S^n is called *free* if there exists an $h : S^n \to S^n$ such that

$$A_i \cap h(A_i) = \emptyset, \qquad \text{for all } i \in \{1, \dots, m\}.$$

The close relation to the antipodal conditions ($h = -\operatorname{id}_{S^n}$) is obvious, nevertheless it turned out that practically nothing of the Theorems (B) and (C) can be preserved with these new conditions. Pannwitz [60] constructed for all $n \in \mathbb{N}$ a free $f : S^n \to \mathbb{R}^2$, and from a slight modification of Theorems 1–3 in [67] one can easily deduce the existence of a free cover $\{A_1, \dots, A_8\}$ of S^n.

B. Starting from the observation that antipodal points on the sphere have distance 2 or geodesic distance π, H. Hopf [40] replaced the antipodal condition by a condition on the (geodesic) distance and proved:

THEOREM 3.2. *Let* $a \in]0,2]$. *Then for each* $f : S^n \to \mathbb{R}^n$ *there exist* $x, y \in S^n$ *with* $| x - y | = a$ *and* $f(x) = f(y)$.

He also obtained a generalization of Theorem C in this context. In fact, these results can easily be deduced from the original antipodal theorem by some clever homotopy argument.

Thus the full antipodal theorem is preserved under this more general condition. Nevertheless some restriction of this statement is necessary as we will see in the following section C.

C. It is quite natural to try to modify Theorem 3.2 in the following way. Instead of a coincidence in two points x and y one would like to obtain a coincidence

$$f(x_1) = \cdots = f(x_k)$$

in k points. Of course, for this one needs an image space of a smaller dimension.

KNASTER'S CONJECTURE (cf. [45]). *Let*

$$f : S^n \to \mathbb{R}^m \quad \text{and} \quad x_1, \ldots, x_{n-m+2} \in S^n, \qquad 2 \leq m \leq n.$$

Does there exist a transformation $g \in O(n+1)$ *such that*

$$f(g\,x_1) = \cdots = f(g\,x_{n-m+2})\,?$$

Of course, $m = n$ is exactly Theorem 3.2. For $m < n$, several partial answers (cf. e.g. [44, 79]) motivate that the dimension assumption is appropriately chosen. Nevertheless, Makeev [55] and Babenko and Bogatjĭ [6] were able to disprove Knaster's conjecture. This makes it clear that only for rather "symmetric" configurations of the points $x_1, \ldots, x_{n-m+2}$ one has a chance for a positive answer.

REMARK. Hopf's condition as in section B is a special case of a so-called multivalued involution.

DEFINITION 3.3. A set $\Gamma \subset X \times X$ is called a *multivalued involution* if

α) for all $x \in X$ there is a $y \in X$ with $(x, y) \in \Gamma$,

β) $(x, y) \in \Gamma$ implies $(y, x) \in \Gamma$.

There is a large number of extensions of the antipodal theorem in terms of multivalued involutions (cf. e.g. [1, 8, 42]).

4. Group Actions

The considerations of the last section are more of theoretical interest rather than of importance for applications. We have seen that one has to be very careful with weakening the antipodal condition. Now in this section we want to have a closer look at the nature of this symmetry condition. We will introduce group actions and equivariant maps, and we will see that they yield an excellent framework to embed the classical antipodal theorem and to extend it in a convincing way. Moreover, this class of generalizations has found various deep applications (cf. section 6).

Let G be a compact Lie group (the only examples of major interest for us will be the finite groups and S^1) and X a topological space.

DEFINITION 4.1. A map

$$\Phi : G \times X \ni (g, x) \longmapsto g\,x \in X$$

is called a *G-action* if

$$g_1(g_2\,x) = (g_1\,g_2)\,x, \qquad \text{for all } g_1, g_2 \in G,\ x \in X$$

and

$$e\,x = x, \qquad \text{for } x \in X,\ e \in G, \text{ the identity of } G.$$

A space X with a G-action is called a *G-space*.

EXAMPLE 1. Let $X = -X \subset \mathbb{R}^n$. We can write the group $\mathbb{Z}/2$ as the multiplicative group $\{-1, +1\}$ of two elements. The canonical map

$$\Phi : \mathbb{Z}/2 \times X \ni (g, x) \longmapsto g\,x \in X$$

is obviously a $\mathbb{Z}/2$-action. The element $-1 \in \mathbb{Z}/2$ induces the antipodal map on X.

DEFINITION 4.2. Let X, Y be G-spaces. A map $f : X \to Y$ with

$$f(g\,x) = g\,f(x), \qquad \text{for all } g \in G,\ x \in X,$$

is called *(G-)equivariant* or a *G-map*.

EXAMPLE 2. Let $X = -X \subset \mathbb{R}^m$ and $Y = -Y \subset \mathbb{R}^n$. An odd map $f : X \to Y$ is a $\mathbb{Z}/2$-map with respect to the antipodal actions as in example 1.

Examples 1 and 2 show that we can easily interpret the classical antipodal theorem in the language of G-spaces and G-maps. In order to obtain the appropriate framework for generalizations in this context, we need some further notations.

DEFINITION 4.3. For a G-space X and a subgroup $H \subset G$ we denote

$$X^H := \{x \in X \ :\ g\,x = x, \text{ for all } g \in H\}.$$

The G-action on X is called

α) *free* if $X^H = \emptyset$ for all subgroups $H \neq \{e\}$,
β) *semifree* if $X^H = X^G$ for all subgroups $H \neq \{e\}$,
γ) *fixed point free* if $X^G = \emptyset$.

Of course, for $\mathbb{Z}/p$-actions with p prime, there is no difference between free and fixed point free actions.

EXAMPLE 3. Typical examples for $\mathbb{Z}/m$-actions on $S^{2n-1}(\subset \mathbb{C}^n)$ are

$$\mathbb{Z}/m \times S^{2n-1} \ni (\alpha, (z_1, \dots, z_n)) \longmapsto (e^{\alpha \frac{2\pi i}{m} k_1} z_1, \dots, e^{\alpha \frac{2\pi i}{m} k_n} z_n) \in S^{2n-1}$$

with given fixed $k_1, \dots, k_n \in \mathbb{Z}$. The action is free iff the greatest common divisor

$$\gcd(k_j, m) = 1, \qquad \text{for } j = 1, \dots, n.$$

The same formula with $m = 1$ and $\alpha \in S^1 = \mathbb{R}/\mathbb{Z}$ also stands for an S^1-action.

EXAMPLE 4. Let X be a topological vector space of 1-periodic functions $f : \mathbb{R} \to \mathbb{R}^n$, i.e.,

$$f(x+1) = f(x), \qquad \text{for all } x \in \mathbb{R},$$

such that for all $t \in \mathbb{R}$,

$$\sigma_t f := f(\cdot + t) \in X, \qquad \text{for all } f \in X,$$

with $\sigma_t : X \to X$ continuous. This operation

$$\mathbb{R} \times X \ni (t, f) \longmapsto \sigma_t f \in X$$

generates an S^1-action on X. It is obvious that, in general, most of the sets X^H for subgroups $H \subset S^1$ are nontrivial.

The pioneers in Borsuk theory for more general group actions were Hirsch [36], Eilenberg [27], and Smith [65] for finite group actions and Hopf and Rueff [41] for S^1-actions. Already in these very early papers of the thirties and forties, it turned out that the more general framework allowed a much larger variety of now not necessarily equivalent extensions of Borsuk's original results. I will concentrate only on those generalizations which turned out to be of major interest for applications or for the theory. These are particularly the extensions of the Theorems A and F.

a) Theorem A says in the language of group actions that if a map $f : S^n \to S^n$ is equivariant with respect to a (special) free $\mathbb{Z}/2$-action on S^n, then $\deg f$ is odd. This can be generalized in the following way.

THEOREM 4.4. *Let G be a compact Lie group acting freely on S^n, and let $f : S^n \to S^n$ be an equivariant map. Then*

$$\deg f \equiv 1 \pmod{o(G)},$$

where $o(G)$ is the order of G.

For infinite groups, this reads $\deg f = 1$.

If G does not act freely on S^n, the situation is much more complicated (cf. [24, 63]), but at least for semifree actions, the generalization is easy to state.

THEOREM 4.5. *Let G be a compact Lie group acting semifreely on S^n through linear isometries. If $f : S^n \to S^n$ is equivariant, then*

$$\deg f \equiv \deg f \mid_{(S^n)^G} \pmod{o(G)}.$$

It may be noted that the linearity of the action is not essential: Smith [65] showed for groups $G = \mathbb{Z}/m$ that $(S^n)^G$ is always a homology sphere. Therefore one has a degree $\deg f \mid_{(S^n)^G}$ as for ordinary spheres and, in fact, the same formula as in Theorem 4.5 holds.

b) Theorem F reads in the language of group actions as follows: For $m < n$, there is no map $f : S^n \to S^m$ which is equivariant with respect to free $\mathbb{Z}/2$-actions on S^n and S^m.

The same result holds for arbitrary compact Lie groups G and free G-actions on S^m and S^n (cf. [25] for the case of finite groups, from which the case of infinite groups follows easily).

The question becomes much more delicate if one allows nonfree G-actions. The case, where the G-actions on S^n or S^m have fixed points, is of no interest, because then the existence of a G-map $f : S^n \to S^m$ implies $(S^m)^G \neq \emptyset$, in which case each constant map $f : S^n \to S^m$ to a point $x \in (S^m)^G$ is G-equivariant.

So let us consider the following question.

Let G be a compact Lie group. For which pairs (m, n) there exist fixed point free G-actions on S^m and S^n and a G-map $f : S^n \to S^m$?

Marzantowicz [56] and Bartsch [11] gave a fairly complete answer to this question (allowing only linear G-actions). It turned out that the answer heavily depends on the structure of the group: For p being a prime number, one obtains in the case of finite groups:

α) If $G = (\mathbb{Z}/p)^k$, then $m \geq n$.

β) If $G = \mathbb{Z}/p^k$, then $m + 1 \geq (n+1)/p^{k-1}$.

γ) G is a p-group$\iff (n \to \infty \Rightarrow m \to \infty)$.

In the case of infinite groups, Bartsch [11] conjectured that p-toral groups (i.e., extensions of a p-group by a torus $(S^1)^k$) replace p-groups in the above equivalence γ).

c) There are also various extensions of the theorems of Borsuk-Ulam (Theorem B) and Lusternik-Schnirelmann-Borsuk (Theorem C) in the context of this chapter for free actions of cyclic groups $\mathbb{Z}/p$ (cf. e.g. [21, 51, 59, 66, 67]). Note the strong (in fact linear) dependance on p in all these results. For example, in the Borsuk-Ulam results, for larger p, one has to choose a smaller dimension m of the image space $\mathbb{R}^m$ in order to obtain the desired coincidences.

5. Index Theories

Up to now, we considered the antipodal theorem and its generalizations only on spheres. But there are very good reasons to go beyond this restriction. Some examples may illustrate this.

EXAMPLE 1. (Bourgin-Yang results [17, 77]) If we replace in Theorem B the assumption $f : S^n \to \mathbb{R}^n$ by $f : S^n \to \mathbb{R}^m$ with $m < n$, we should expect a larger set of coincidence points. How can we measure its size?

EXAMPLE 2. (Clark [20]) Let $V \subset \mathbb{R}^n$ be an m-dimensional linear subspace and $A = -A \subset \mathbb{R}^n$. Clark showed that if A is *big enough* then $A \cap V \neq \emptyset$. Of course we have to make precise what we mean with *big enough*.

EXAMPLE 3. (Variational methods) Suppose $f : S^n \to \mathbb{R}$ is an even C^1-functional. It is rather evident (deformation lemma) that a $c_0 \in \mathbb{R}$, where the topological structure of the spaces

$$f_c := f^{-1}(]-\infty, c])$$

changes, must be a critical value of f, i.e.,

$$f'(x) = 0, \qquad \text{for some } x \in f^{-1}(c_0).$$

(It is very instructive to check this by drawing the graph of e.g. some polynomial on $\mathbb{R}$.) It is important to find a practicable criterion for this change in the topological structure.

In all these examples, index theories turned out to be an appropriate tool. What is an index theory?

Let X be a free G-space (G a compact Lie group) and $A \subset X$ a G-invariant subset. An integer–valued index

$$\operatorname{ind} A \in \mathbb{N} \cup \{0, \infty\}$$

has the following properties:

(i) $\operatorname{ind} A = 0 \iff A = \emptyset$.

(ii) $f : A_1 \to A_2$ equivariant $\implies \operatorname{ind} A_1 \leq \operatorname{ind} A_2$.

(iii) $A \subset X$ closed invariant $\implies$ there exists an invariant neighborhood B of A with $\operatorname{ind} B = \operatorname{ind} A$.

(iv) $\operatorname{ind} A \cup B \leq \operatorname{ind} A + \operatorname{ind} B$.

(v) $\operatorname{ind} S^n \approx \alpha(n+1)$ with some $\alpha \in]0, 1]$ depending only on G.

Before sketching some possible ways of constructing such an index theory, I would like to indicate its relation to Borsuk's theory.

In case of $G = \mathbb{Z}/2$, for the standard index theories one even has

$$\operatorname{ind} S^n = n + 1,$$

at least

$$\operatorname{ind} S^m < \operatorname{ind} S^n \qquad \text{if } m < n.$$

The condition (ii) is nothing but an extension of Theorem F.

There are mainly two techniques for the construction of an index, which, in fact, yield different index theories.

Genus or B-Index. (cf. [47, 69, 78]) The idea of this definition is a comparison with free G-spaces of a particularly simple structure: Let X be a free G-space. Then the genus $g(X, G)$ is defined by

$$g(X, G) := \inf\{n \in \mathbb{N}_0 \,:\, \exists \text{ equivariant } f : X \to G^{(n)}\},$$

where $G^{(n)}$ is the n-fold join $G * \cdots * G$ with the canonical (diagonal) G-action. If no equivariant f exists whatever n may be, we understand the definition as $g(X, G) = \infty$, whereas $g(X, G) = 0$ for $X = \emptyset$.

In the special case of $G = \mathbb{Z}/2$, there is a canonical identification

$$(\mathbb{Z}/2)^{(n)} \cong S^{n-1},$$

which is equivariant with respect to the standard $\mathbb{Z}/2$-actions on $(\mathbb{Z}/2)^{(n)}$ and S^{n-1}. A similar remark holds true in case of $G = S^1$ since

$$(S^1)^{(n)} \cong S^{2n-1}.$$

In fact, the genus for $\mathbb{Z}/2$- and S^1-spaces was first introduced via equivariant embeddings into spheres. Cf. also [66] for $\mathbb{Z}/p$-spaces. I should also mention that in case of finite groups G there is another characterization of the genus via coverings, which is sometimes quite useful (cf. [66]).

Cohomological Index. (cf. [28, 29, 43]) We have seen that the idea of the genus is very simple and elementary, and, in fact, this notion is usually very easy to apply. Nevertheless, sometimes this index theory has some shortcomings. In particular, it is a nontrivial task to find good estimates for the genus of specific G-spaces. Furthermore, it turned out that the genus is not a good basis for the extension to more general situations, in particular to nonfree G-spaces.

Here other index theories have shown to be much more flexible, namely cohomological index theories. Just to give some flavor of this notion, I will give a very short sketch of the definition of an ideal–valued and a numerical–valued cohomological index theory. I would suggest to try to interpret the cohomological proof of Theorem G from the point of view of this definition.

We assume X to be a free paracompact G-space, and let EG be a universal G-space and $BG := EG/G$ its orbit space (*classifying space*). With the Borel cohomology H_G^* with coefficients in some field (e.g. the rationals or $\mathbb{Z}/2$), we can define the ideal–valued index as

$$\operatorname{Ind}^G X := \ker(\varphi^* : H_G^*(EG) \to H_G^*(X)),$$

with any equivariant map $\varphi : X \to EG$, or as

$$\operatorname{Ind}^G X := \ker(c^* : H_G^*(\cdot) \to H_G^*(X)),$$

with the constant map $c : X \to \cdot$ to the one point space $\cdot$. With the induced map $\bar{\varphi} : X/G \to BG$ on orbit spaces, these definitions are also equivalent to

$$\operatorname{Ind}^G X := \ker(\bar{\varphi}^* : H^*(BG) \to H^*(X/G)).$$

Of course, for this ideal–valued index, one has to reinterpret the defining properties of an index theory. E.g. property (iv) now reads

$$\operatorname{Ind}^G A \cdot \operatorname{Ind}^G B \subset \operatorname{Ind}^G A \cup B.$$

One obtains a numerical–valued index theory by

$$\operatorname{ind}^G X := \mid \operatorname{Ind}^G X \mid := \dim(H_G^*(\cdot)/\operatorname{Ind}^G X).$$

6. Applications

I only want to sketch the broad spectrum of applications in many mathematical fields which shows the fundamental role of the antipodal theorem and its generalizations. For a more complete collection of applications cf. [68].

Invariance of Domain. In one of the classical applications, one shows that continuous and injective maps $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ (U open) are open, a result which can easily be generalized to locally compact perturbations of the identity in Banach spaces [16, 32].

Measure Theory. The famous ham-sandwich theorem says that, given bounded measurable sets $M_1, \dots, M_n \subset \mathbb{R}^n$, there is a hyperplane P in $\mathbb{R}^n$ which simultaneously bisects each of the sets M_i such that the two parts of M_i have the same measure (cf. [26]).

Algebra. Dai, Lam, and Peng [23] (cf. also [22]) used the Borsuk-Ulam theorem for a proof of the fact that the quotient

$$\mathbb{R}[x_1, \dots, x_n]/(1 + x_1^2 + \dots + x_n^2)$$

has level n, i.e., -1 cannot be written as a sum of less than n squares.

Graph Theory. Bárány [9] and Lovász [50] proved Kneser's conjecture: Split the set of n-subsets of $\{1, \dots, 2n+k\}$ in $k+1$ classes, then one of them contains two disjoint sets. Another combinatorial application of the Borsuk-Ulam theorem is the nice result on *splitting necklaces* [3, 4].

The Van Kampen-Flores Theorem. This result says that the n-skeleton of the simplex Δ_{2n+2} does not embed into $\mathbb{R}^{2n}$ [31, 35].

Banach Spaces. Voigt [72] used a variant of the antipodal theorem (cf. [30, 72]) to show that if L is a linear subspace of the space $CBV[0,1]$ of continuous functions with bounded variation such that L is closed in $C[0,1]$, then $\dim L < \infty$.

Differential Equations. Many existence problems for differential equations (mostly boundary value problems or periodic problems) are of the abstract structure

$$(f + L)x = y,$$

with an odd f and a linear L, or of the structure

$$(g + L)x = 0,$$

where again L is linear and g is asymptotically odd for x large. Under suitable assumptions, both cases can be reduced to Theorem A or some infinite dimensional generalization.

Variational Methods. The techniques which we sketched in example 3 of section 5 yield results on the existence and number of critical values of even (or more generally G-invariant, e.g. S^1-invariant) functionals. Via the Euler-Lagrange formalism, quite often critical points are directly related to solutions of differential equations. Concerning S^1-invariant functionals, there are very nice applications to problems of estimating the number of periodic solutions of Hamiltonian systems.

For specific references on applications to differential equations and variational methods cf. [57, 68, 80].

REFERENCES

[1] M. K. AGOSTON, *Generalized antipodes and the Borsuk antipode theorem*, Comment. Math. Helv. **46** (1971), 457–466.

[2] J. C. ALEXANDER AND J. A. YORKE, *The homotopy continuation method: numerically implementable topological procedures*, Trans. Amer. Math. Soc. **242** (1978), 271–284.

[3] N. ALON, *Splitting necklaces*, Adv. in Math. **63** (1987), 247–253.

[4] N. ALON AND D. B. WEST, *The Borsuk-Ulam theorem and bisection of necklaces*, Proc. Amer. Math. Soc. **98** (1986), 623–628.

[5] J. K. ARASON AND A. PFISTER, *Quadratische Formen über affinen Algebren und ein algebraischer Beweis des Satzes von Borsuk-Ulam*, J. Reine Angew. Math. **331** (1982), 181–184.

[6] I. K. BABENKO AND S. A. BOGATYĬ, *Mapping a sphere into Euclidean space*, Mat. Zametki **46** (1989), no. 3(126), 3–8 (Russian); English transl. in Math. Notes **46** (1989), 683–686.

[7] P. BACON, *Equivalent formulations of the Borsuk-Ulam theorem*, Canad. J. Math. **18** (1966), 492–502.

[8] E. G. BAJMÓCZY AND I. BÁRÁNY, *On a common generalization of Borsuk's and Radon's theorem*, Acta Math. Acad. Sci. Hungar. **34** (1979), 347–350.

[9] I. BÁRÁNY, *A short proof of Kneser's conjecture*, J. Combin. Theory Ser. A **25** (1978), 325–326.

[10] ———, *Borsuk's theorem through complementary pivoting*, Math. Programming **18** (1980), 84–88.

[11] T. BARTSCH, *On the existence of Borsuk-Ulam theorems*, Topology **31** (1992), 533-543.

[12] P. BOHL, *Über die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage*, J. Reine Angew. Math. **127** (1904), 179–276.

[13] K. BORSUK, *Sur les rétractes*, Fund. Math. **17** (1931), 152–170.

[14] ———, *Über die Zerlegung einer euklidischen n-dimensionalen Vollkugel in n Mengen*, Verhandlungen des Internationalen Mathematiker–Kongresses Zürich 1932, II. Band: Sektionsvorträge, Orel Füssli, Zürich, 1932, pp. 192–193.

[15] ———, *Drei Sätze über die n-dimensionale Sphäre*, Fund. Math. **20** (1933), 177–190.

[16] ———, *Über stetige Abbildungen der euklidischen Räume*, Fund. Math. **21** (1933), 236–243.

[17] D. G. BOURGIN, *On some separation and mapping theorems*, Comment. Math. Helv. **29** (1955), 199–214.

[18] L. E. J. BROUWER, *Über Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1912), 97–115.

[19] A.L. BROWN, *The Borsuk-Ulam theorem and orthogonality in normed linear spaces*, Amer. Math. Monthly **86** (1979), 766–767.

[20] D. C. CLARK, *A variant of the Lusternik-Schnirelmann theory*, Indiana Univ. Math. J. **22** (1972), 65–74.

[21] F. COHEN AND J. E. CONNETT, *A coincidence theorem related to the Borsuk-Ulam theorem*, Proc. Amer. Math. Soc. **44** (1974), 218–220.

[22] Z. D. DAI AND T. Y. LAM, *Levels in algebra and topology*, Comment. Math. Helv. **59** (1984), 376–424.

[23] Z. D. DAI, T. Y. LAM AND C. K. PENG, *Levels in algebra and topology*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), 845–848.

[24] E. N. DANCER, *Symmetries, degree, homotopy indices and asymptotically homogeneous problems*, Nonlinear Anal. **6** (1982), 667–686.

[25] A. DOLD, *Simple proofs of some Borsuk-Ulam results*, Proceedings of the Northwestern Homotopy Theory Conference (Evanston, IL., 1982), Contemp. Math., vol. 19, Amer. Math. Soc., Providence, 1983, pp. 65–69.

[26] J. DUGUNDJI AND A. GRANAS, *Fixed point theory*, vol. I, Monografie Matematyczne, Polish Scientific Publishers (PWN), Warsaw, 1982.

[27] S. EILENBERG, *On a theorem of P. A. Smith concerning fixed points for periodic transformations*, Duke Math. J. **6** (1940), 428–437.

[28] E. FADELL, *Ideal-valued generalizations of Ljusternik-Schnirelmann category, with applications*, Topics in Equivariant Topology, Variational Methods in Nonlinear Problems,

(Montreal, 1986) Sémin. Math. Supér., Sémin. Sci. OTAN (NATO Adv. Study Inst.), vol. 108, 1989, pp. 9–52.

[29] E. R. FADELL AND P. H. RABINOWITZ, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. **45** (1978), 139–174.

[30] KY FAN, *A generalization of Tucker's combinatorial lemma with topological applications*, Ann. of Math. (2) **56** (1952), 431–437.

[31] A. FLORES, *Über n-dimensionale Komplexe, die im $\mathbb{R}_{2n+1}$ absolut selbstverschlungen sind*, Ergeb. Math. Kolloq. **6** (1935), 4–7.

[32] A. GRANAS, *On continuous mappings of open sets in Banach spaces*, Bull. Acad. Polon. Sci. **6** (1958), 25–29. (Russian; English summary)

[33] ———, *The theory of compact vector fields and some of its applications to topology of functional spaces. I*, Rozprawy Mat., vol. 30, Polish Scientific Publishers (PWN), Warsaw, 1962.

[34] W. GROMES, *Ein einfacher Beweis des Satzes von Borsuk*, Math. Z. **178** (1981), 399–400.

[35] B. GRÜNBAUM, *Convex polytopes*, Interscience Publishers, London–New York–Sydney, 1967.

[36] G. HIRSCH, *Une généralization d'un théorème de M. Borsuk concernant certaines transformations de l'analysis situs*, Acad. Roy. Belgique. Bull. Cl. Sci. (5) **23** (1937), 219–225.

[37] H. HOPF, *Abbildungsklassen n-dimensionaler Mannigfaltigkeiten*, Math. Ann. **96** (1927), 209–224.

[38] ———, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann. **104** (1931), 637–665.

[39] ———, *Freie Überdeckungen und freie Abbildungen*, Fund. Math. **28** (1937), 33–57.

[40] ———, *Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze*, Portugal. Math. **4** (1943/44), 129–139.

[41] H. HOPF AND M. RUEFF, *Über faserungstreue Abbildungen der Sphären*, Comment. Math. Helv. **11** (1938/39), 49–61.

[42] J. W. JAWOROWSKI, *Theorem on antipodes for multi-valued mappings and a fixed point theorem*, Bull. Acad. Polon. Sci. **4** (1956), 187–192.

[43] ———, *A Borsuk-Ulam theorem for $O(m)$*, Topics in Equivariant Topology, Variational Methods in Nonlinear Problems, (Montreal, 1986) Sémin. Math. Supér., Sémin. Sci. OTAN (NATO Adv. Study Inst.), vol. 108, 1989, pp. 105–118.

[44] S. KAKUTANI, *A proof that there exists a circumscribing cube around any bounded closed convex set in $\mathbb{R}^3$*, Ann. of Math. (2) **43** (1942), 739–741.

[45] B. KNASTER, *Problème 4*, Colloq. Math. **1** (1948), 30–31.

[46] M. KNEBUSCH, *An algebraic proof of the Borsuk-Ulam theorem for polynomial mappings*, Proc. Amer. Math. Soc. **84** (1982), 29–32.

[47] M. A. KRASNOSEL'SKIĬ, *On the estimation of the number of critical points of functionals*, Uspehi Mat. Nauk **7** (1952), no. 2 (48), 157–164. (Russian)

[48] M. G. KREĬN, M. A. KRASNOSEL'SKIĬ AND D. P. MIL'MAN, *On the defect numbers of linear operators in a Banach space and some geometrical questions*, Sb. Trudov Inst. Mat. Akad. Nauk Ukrain. SSR no. **11** (1948), 97–112. (Russian)

[49] S. LEFSCHETZ, *Introduction to topology*, Princeton University Press, Princeton, 1949.

[50] L. LOVÁSZ, *Kneser's conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A **25** (1978), 319–324.

[51] E. L. LUSK, *The mod p Smith index and a generalized Borsuk-Ulam theorem*, Michigan Math. J. **22** (1975), 151–160.

[52] L. LUSTERNIK, *Sur quelques méthodes topologiques dans la géometrie différentielle*, Atti del congresso internazionale dei matematici (Bologna, 1928), tomo IV, Nicola Zanichelli, Bologna,. 1931, pp. 291–296.

[53] L. LUSTERNIK AND L. SCHNIRELMANN, *Topological methods in variational problems*, Issledowatelskiĭ Institut Matematiki i Mechaniki pri O. M. G. U., Moscow, 1930. (Russian)

[54] ———, *Méthodes topologiques dans les problèmes variationnels*, Hermann & Cie., Paris, 1934.

[55] V. V. MAKEEV, *Some properties of continuous mappings of spheres and problems in combinatorial geometry*, Geometric Questions in the Theory of Functions and Sets, Kalinin. Gos. Univ.,, Kalinin, 1986, pp. 75–85. (Russian)

[56] W. MARZANTOWICZ, *An almost classification of compact Lie groups with Borsuk-Ulam properties*, Pacific J. Math. **144** (1990), 299–311.

[57] J. MAWHIN AND M. WILLEM, *Critical point theory and Hamiltonian systems*, Springer-Verlag, Berlin, 1989.

[58] M. D. MEYERSON AND A. H. WRIGHT, *A new and constructive proof of the Borsuk-Ulam theorem*, Proc. Amer. Math. Soc. **73** (1979), 134–136.

[59] M. H. A. NEWMAN, *Fixed point and coincidence theorems*, J. London Math. Soc. **27** (1952), 135–140.

[60] E. PANNWITZ, *Eine freie Abbildung der n-dimensionalen Sphäre in die Ebene*, Math. Nachr. **7** (1952), 183–185.

[61] H. POINCARÉ, *Sur les courbes définies par les équations différentielles*, J. Math. Pures Appl. (4) **2** (1886), 151–217.

[62] P. H. RABINOWITZ, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math. **3** (1973), 161–202.

[63] R. RUBINSZTEIN, *On the equivariant homotopy of spheres*, Dissertationes Math. (Rozprawy Mat.), vol. 134, Polish Scientific Publishers (PWN), Warsaw, 1976.

[64] L. SCHNIRELMANN, *Über eine neue kombinatorische Invariante*, Monatsh. Math. Phys. **37** (1930), 131–134.

[65] P. A. SMITH, *Fixed points of periodic transformations*, Appendix B in Algebraic topology by S. Lefschetz, Amer. Math. Soc., New York, 1942, pp. 350–373.

[66] H. STEINLEIN, *Borsuk-Ulam Sätze und Abbildungen mit kompakten Iterierten*, Dissertationes Math. (Rozprawy Mat.), vol. 177, Polish Scientific Publishers (PWN), Warsaw, 1980.

[67] ———, *On the theorems of Borsuk-Ulam and Ljusternik-Schnirelmann-Borsuk*, Canad. Math. Bull. **27** (1984), 192–204.

[68] ———, *Borsuk's antipodal theorem and its generalizations and applications: a survey*, Méthodes topologiques en analyse non linéaire, (Montreal, 1983), Sémin. Math. Supér., Sémin. Sci. OTAN (NATO Adv. Study Inst.), vol. 95, 1985, pp. 166–235.

[69] A. S. ŠVARC, *The genus of a fibre space*, Trudy Moscov Mat. Obšč. **10** (1961), 217–272; **11** (1962), 99–126 (Russian); English translation in Amer. Math. Soc., Translat., II. Ser. **55** (1966), 49–140.

[70] M. J. TODD AND A. H. WRIGHT, *A variable dimension simplicial algorithm for antipodal fixed-point theorems*, Numer. Funct. Anal. Optim. **2** (1980), 155–186.

[71] A. W. TUCKER, *Some topological properties of disk and sphere*, Proc. First Canadian Math. Congress (Montreal, 1945), University of Toronto Press, Toronto, 1946, pp. 285–309.

[72] J. VOIGT, *On Y-closed subspaces of X, for Banach spaces $X \subset Y$; existence of alternating elements in subspaces of $C(J)$*, Pacific J. Math. **81** (1979), 253–266.

[73] P. VOLKMANN, *Ein Störungssatz für abgeschlossene lineare Operatoren*, Math. Ann. **234** (1978), 139–144.

[74] J. W. WALKER, *A homology version of the Borsuk-Ulam theorem*, Amer. Math. Monthly **90** (1983), 466–468.

[75] F. WILLE, *Zwei Überdeckungssätze für die n-dimensionale Kugel*, Elem. Math. **25** (1970), 49–54.

[76] ———, *Analysis*, Eine anwendungsbezogene Einführung, B. G. Teubner, Stuttgart, 1976.

[77] C.-T. YANG, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson, I*, Ann. of Math. (2) **60** (1954), 262–282.

[78] ———, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson, II*, Ann. of Math. (2) **62** (1955), 271–283.

[79] ———, *On maps from spheres to Euclidean spaces*, Amer. J. Math. **79** (1957), 725–732.

[80] E. ZEHNDER, *Periodische Lösungen von Hamiltonschen Systemen*, Jahresber. Deutsch. Math.–Verein. **89** (1987), 33–59.

HEINRICH STEINLEIN
Mathematisches Institut, Universität München
München